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NUMBERS, VARIABLES AND MR. RUSSELL'S PHILOSOPHY.

PHILOSOPHERS not infrequently take mathematics as a field for the exercise of their fancies. If the starting-point be a really sound philosophy, the result of an excursion into mathematics enures to the benefit of both disciplines. If, however, the start be a metaphysics or a logic that is essentially erroneous, the results attained may be novel or startling, but have no place in the body of our scientific knowledge. To say that the investigations of any particular philosopher comes under this second head seems invidious, yet it is necessary to so stigmatize the false systems if the way is to be kept open for the true. Of all the impediments to the cultivation of mathematical science on a philosophical basis there can be none greater than the putting forth a pseudo-philosophy in the guise of true doctrine. As such an impediment do we esteem the work of Mr. Bertrand Russell, who is a recent writer of some repute, though whether he ought to be classed as philosopher who devotes himself to mathematics or as a mathematician who dips into philosophy is a moot question.

Mr. Russell's labors seem to have as their burden the reorganization of mathematics upon the basis of what he is pleased to term "one of the greatest discoveries of our age"; a discovery, if such it be, largely due to Mr. Russell himself (not to speak of his forerunner, Professor Peano) "that all Mathematics is Symbolic Logic."[1] To encounter such a statement is rather startling to any one who has been accustomed to class mathematics among the deduc-

[1] *Principles of Mathematics,* Cambridge, at the University Press, Vol. I, 1913, p. 5.

tive sciences. Symbolic logic is merely deductive logic treated in a particular way, and deductive logic is not usually understood to take within its scope any special deductive science, but is supposed merely to give an account of the methods of deduction employed in these sciences. The view taken by Mr. Russell is tenable only when mathematics and logic are understood to have scopes different from those usually accepted for them. Indeed the mathematical science developed by the school of Peano and Russell has some radical differences, not merely in scope but also in method, from what is ordinarily expounded under the name of mathematics. These neo-mathematicians hold that mathematics ought not to follow the method of laying down a set of special mathematical axioms and postulates for each branch of the science and deducting therefrom the theorems of that branch. The only principles Mr. Russell would have pure mathematics put forword are "ten principles of deduction and ten other premises of a general logical nature."[2] The natural result is that what Mr. Russell attains are not the theorems of ordinary mathematics. Where ordinary mathematics would deduce from the axioms A, B and C the theorem T, Mr. Russell is satisfied to have mathematics not assert A or B or C at all—much less T—but merely prove and assert the proposition that "A, B and C imply T"; in other words that the theorem T is a logical consequence of A, B and C if these are given as premises.

To replace the ordinary theorems of mathematics by propositions of implication is alone insufficient to bring mathematics into the realm of symbolic logic. Mr. Russell goes still further and so extends the denotations of the mathematical symbols made use of that the propositions of pure mathematics become the merest of shells. A mathematical formula where x and y originally meant quantities

[2] P. 4.

is by Mr. Russell so extended that x and y may have as wide a range of applicability as the X and Y of the canonical logical proposition: "Every X is a Y." Thus if by some far-fetched interpretation of addition, multiplication etc. the formula $(x+y)^2 = x^2 + 2xy + y^2$ can be made to yield a significance when x and y instead of meaning quantities are understood to mean Plato and Socrates respectively, Mr. Russell would bring the formula as thus construed within the mathematical field. Pure mathematics then, with Mr. Russell, is characterized, not by the import of its propositions, but by their form, and its end in each question it takes up is merely to find what form of proposition is implied by a set of propositions of specified forms. It thus becomes identified with symbolic logic provided deductive logic be held to deal, not merely with generalities in the theory of inference, but to have an essential part of its scope the consideration of every possible combination of the types of propositions which it accepts, and to endeavor in each such case to find what form of proposition is to be taken as the type of conclusion inferable from this combination of premises.

We shall not here debate the question of how far this new view of the scope of pure mathematics is worthy of being called a "great discovery." But we must point out that Mr. Russell is not entirely consistent in his adhesion to this view. Under it either pure mathematics must be completely identified with symbolic logic or must be put in the rank of a subdivision of the latter discipline. No other alternative is reconcileable with the statement that "all Mathematics is Symbolic Logic." Mr. Russell however, without seeing the need for first retracting this statement, is led by "respect for tradition" and "desire to adhere to usage" to draw a distinction under which logic "consists of the premises of mathematics" together with certain other propositions not mathe-

matical, while mathematics "consists of all the consequences of the above premises" together with some, but not all, of these premises themselves.[3] Obviously then, mathematics will cover a field that is not included in logic at all, and the mathematics of this field can certainly not be symbolic logic.

Notwithstanding the purely formal rôle that Mr. Russell would assign to pure mathematics, he takes up in his *Principles* various questions that seem to pertain to the matter rather than to the form of mathematical science. After enumerating certain "indefinables of mathematics" he proceeds to the definables, and begins by the discussion of number. The doctrine of number is fundamental in mathematical philosophy; let us see how Mr. Russell handles it.

Numbers, Mr. Russell tells us, are "applicable essentially to classes."[4] This word *class* is a favorite one with Mr. Russell, and he often uses it where another word would be more appropriate. In the present case *group* might well be adopted in preference to class. For ordinarily, when reference is made to a class of objects, what is in mind is something about objects of that class taken individually—not about the objects as constituting a collectivity. Now (except when there is but a single object at hand, a case which gives rise to just as much difficulty when we speak of "class" as when we use the word "group") it is precisely this idea of collectivity, so aptly suggested by *group,* that is in evidence when we speak of number. The number of objects in a group is a number that belongs to the group as a whole, not to any of its objects taken separately.

A number then is something belonging to a group—we replace "applicable to" by "belonging to" without stopping to comment on the impropriety of Mr. Russell's use

[3] P. 9. [4] P. 112.

of the former phraseology—and if we follow Mr. Russell we must regard it as a property of that group. "Numbers," he tells us, "are to be regarded as properties of classes."[5] Presumably Mr. Russell does not use "property" in the sense of the old *proprium*, a name by which certain attributes were distinguished from the essential on the one hand and the accidental on the other; in this sense the statement would not be true, for to have a number is of the very essence of a group. Taking, however, property as a mere synonym of attribute, and substituting group for class, no fault need be found with what Mr. Russell says. A number is truly an attribute of a group of objects. But Mr. Russell too hastily proceeds to inquire "Under what circumstances do two classes have the same number?"[6] when the next question ought in fact to be: Do two groups ever really have the *same* number? In a mere mathematician it is pardonable to be unaware that the question of identity of attributes of distinct objects, or distinct groups of objects, is a debatable one, but surely any writer who aspires to be ranked as a philosopher ought to know that eminent thinkers have been at variance in this matter. Thus, going back only a few years, we find that a brief but very interesting discussion took place between Mill and Spencer as to whether two different objects could be said to have the same attribute.

The point really at issue is this: when one speaks of sameness (or identity) in such a case, is he so using language as to mark all the distinctions that ought to be made, or is he ignoring some of them, is he promoting clearness of thought and speech, or is his phraseology pregnant with obscurity and confusion? This question never seems to have occurred to Mr. Russell, though it arises whether the attributes are qualities belonging to different individuals or are number attributes (quantities) belonging to

[5] P. 113. [6] *Ibid.*

different groups. The primary use of *same* or *identical* in connection with attributes is in the case where an object is viewed continuously for a time by an observer who perceives a certain attribute of that object to undergo no change—to be the *same* at the end as at the beginning of the observation. It is in this primary sense of "same" that we speak of the color of an object *remaining the same.*[7] Quite a different case however is at hand when an observer compares two distinct bodies which are before him and decides that they are exactly alike in color, or compares two distinct groups of objects and decides that they are exactly alike (equal) as to number. Colloquially, it is true, "same" would be used here as well as in the previous case; the two bodies would be said to have the same color, and the two groups to contain the same number of objects, but is this colloquial use of "same" worthy of a philosopher? Is it at all suited to the requirements of an exact science? "Same" in its primary sense is used to express one set of facts concerning attributes; why should it be also used to express facts of entirely different character when there are at hand other words, *like* and *equal,* perfectly adapted to convey this second sense? To use "same" and "identical" in this sense is as absurd and misleading as it would be to call two houses exactly alike the same house.

It would be a great mistake to regard the distinction between identity and equality of numbers as a mere verbal subtlety. True, it is not of any moment so far as computation is concerned; it does not affect the compilation of

[7] Under this first head likewise comes the case in which sameness of attribute is asserted where the asserter has not given continuous attention to the object, having viewed it only at the beginning and at the end of an interval of time, but intends to assert that *if* the observation had been continuous no change would have been noted by the observer. A valid claim to classification under another head may however be granted to the case in which change takes place but the ultimate result is precisely the original state of affairs; where, for instance, a body changes color but finally takes on a color attribute exactly like the one it originally possessed—otherwise put, "returns to the same color."

the logarithmic tables used by the engineer or have any bearing on the numerical calculations by which an astronomer predicts an eclipse. But in the philosophy of mathematics the results of this distinction are far reaching. To begin with, it shows us that 1, 2, 3, etc. are general names like "man" and "animal," and not individual names like "Socrates" and "Plato." Mr. Russell's assertion: "It is plain that we cannot take the number 1 itself twice over, for there is one number one and there are not two instances of it,"[8] is quite untenable. Each of the names 1, 2, 3, etc. is a class name belonging to a class containing many numbers, and we may legitimately speak of a one, a two, a three, etc. instead of adhering to the customary locution which omits the article. In none of these classes, which may most aptly be called *value classes,* is any member identical with another member; thus no three is the same as any other three, though both have the same class name, just as both Socrates and Plato belong to the same class, *man,* and alike have "man" as class name.[9]

Since numerals are general names, it follows that such an equation as $5 = 3 + 2$ is not a singular proposition analogous to "Plato was a contemporary of Socrates," but is a universal proposition analogous to "Every man is an animal." It is an assertion about every five, but should not be read: "Every five is equal to every sum of a three plus a two," for each sum of a three and a two is a five, and though equal to every other five cannot be equal to itself. We must read the equation: "Every five is equal to each sum of a three plus a two unless this sum is itself the five in question," or, better, recognize plainly that the so-called

[8] P. 118.

[9] A vague appreciation of the distinction which ought to be drawn between identity and the belonging to the same class appears in the mathematical use of the word *value.* Mathematicians sometimes speak of two numbers as having the same value; a useful and convenient phraseology if "having the same value" means, not identical, but belonging to the same class, while it is a quite inexcusable circumlocution if what be meant is merely that the "two" numbers are identical—are not two numbers but one and the same number.

"sign of equality" does not concern equality alone but has identity as an alternative possibility, and read our equation as "Every five is equal to or identical with each sum of a three plus a two." That =, as used in mathematics, has commonly two alternatives in view seems to have escaped the eyes of mathematicians. Usually it is read as "equals"; Cayley defines an equation to be "an expression or statement of the equality of two quantities."[10] Schroeder however says that "If a and b are any two names for the same number, we write $a = b$. A proposition of this kind is called an equation."[11] According to this, an equation would always have identity in view, and = would never concern equality at all! Weber is likewise wrong in his definition, telling us that an equation is "a proposition which expresses that a symbol a has the same significance as another symbol b, which we express in mathematical symbolism $a = b$."[12] For, to take the above case of $5 = 3 + 2$, it is not true that 5 has the same significance as $3 + 2$ any more than man has the same significance as mathematician. Every sum of a three plus a two is a five, but not every five has been brought into existence by the addition of a two to a three. Recognition of the true doctrine of number will be found to throw a much-needed light, not only upon the interpretation of equations, but also upon many other matters that mathematics has hitherto left in obscurity. We cannot however stop to dwell on the various ramifications of this doctrine, but must pass on to a further consideration of Mr. Russell's philosophy.

Mr. Russell decides that two groups ("classes") have "the same number" when their members can be correlated in a one-to-one relation, and so defines a one-to-one relation as to enable him to include even the case of groups which comprise nothing at all and give rise to the number zero.

[10] *Collected Mathematical Papers,* Vol. II, Art. "Equation."
[11] *Lehrbuch der Arithmetik und Algebra,* Vol. I, p. 23.
[12] *Encyklopädie der elementaren Algebra und Analyse,* p. 17.

He is not satisfied however to rest here, or with the dictum of "Peano and common sense"[13] that when there subsists such a relation (a relation specified as reflexive, symmetrical and transitive) the two groups have a common property called their number. This definition of numbers "by abstraction" is inadequate, he holds, because nothing is laid down by it which would logically bar there being common to the groups many different properties each answering the description, and thus a number is not by the definition uniquely distinguished among the various properties that groups may have in common. Mr. Russell prefers to give what he calls a nominal definition. Premising first that two groups are to be termed *similar* if they can have their objects put into a one-to-one correspondence, and that a group is to be regarded as similar to itself, he defines the number of a group ("class") as "the class of all classes similar to the given class."[14] This is assuredly the most remarkable definition of a number that has ever been penned. It is as though one would define whiteness as the class of all white objects. Mr. Russell himself, though he characterizes it as "an irreproachable definition of the number of a class in purely logical terms,"[14] admits that his definition appears at first sight to be a wholly indefensible paradox, and his attempt to defend it puts it in no more favorable light. He informs us that "when we remember that a class-concept is not itself a collection, but a property by which a collection is defined, we see that if we define the number as the class-concept, not the class, a number is really defined as a common property of a set of similar classes and of nothing else,"[14] but, even were it true that a number was such a property, this contention would be of no avail to palliate the faults of his definition in which the number *is* defined as the class.[15]

[13] P. 114. [14] P. 115.

[15] On page 131 he tells us that "a class cannot be identified with its class-

What does Mr. Russell really mean? On careful consideration we are impelled to conclude that he is merely indulging in what seems to be his favorite vice—confusion between matters that are in fact quite distinct. As justification of his definition, he tells us that such a word as *couple* "obviously does denote a class of classes." The true facts are that "couple" is a class name, and the class to which it belongs is composed of groups. A couple however is not the class of these groups—it is not the totality of couples. If Mr. Russell wished to define the name *totality of couples* or *all couples* his definition of a number would not be inappropriate. But the totality of couples is not a number.[16] A couple taken alone may be termed a number: a concrete number whose number attribute is an abstract number, a two. Mr. Russell makes however no distinction between concrete numbers and abstract numbers; between a group of objects and the number attribute which belongs to that group (and which for distinctiveness had best be called a *natural number*). He seems on the one hand to think that the definition by abstraction, which defines a number as a property common to certain groups, has reference to precisely the same sort of numbers as a definition framed for such numbers as a couple or a trio, while on the other hand he apparently labors under the delusion that a couple is the same thing as the totality of all couples, a trio the same thing as the totality of all trios, etc. He would not, we presume, contend that a soldier is the same thing as a regiment, but he takes a stand which is just as untenable.

Before going any further into questions that belong to mathematics, properly speaking, we must turn back to

concept." We need hardly say that in describing a class-concept as a *property* Mr. Russell is regarding concepts in a light hitherto unknown to philosophy.

[16] At least it is not such a number (a couple) as Mr. Russell has in view. Of course, if the totality of couples be taken as a group whose members are couples, this group, which will comprise millions of members, has just as much right to be regarded as a concrete number as a group composed of two individual members.

consider Mr. Russell's treatment of certain matters that are purely philosophical. Let us begin with what he calls *terms.* "Whatever may be an object of thought," says Mr. Russell, "or may occur in any true or false proposition, or can be counted as *one,* I call a *term.* This then is the widest word in the philosophical vocabulary. I shall use as synonymous with it the words unit, individual, and entity. The first two emphasize the fact that every term is *one,* while the third is derived from the fact that every term has being, i. e., *is* in some sense. A man, a moment, a class, a relation, a chimera, or any thing else that can be mentioned, is sure to be a term; and to deny that such and such a thing is a term must always be false."[17] This "widest word in the philosophical vocabulary" is not however really the widest, it seems, for only twelve pages later Mr. Russell tells us in an inobtrusive note: "I shall use the word *object* in a wider sense than *term,* to cover both singular and plural, and also cases of ambiguity, such as 'a man.' The fact that a word can be framed with a wider meaning than *term* raises grave logical problems." This retraction of what was first said, coming so soon, does not tend to make us believe that Mr. Russell's original remarks concerning "terms" were based upon long and profound thought. And in fact an examination shows Mr. Russell's view of the matter to be very far from what may be demanded of a philosopher.

To survey the question properly we had best begin with names, or rather with substantive words and phrases, and these for purposes of philosophy we may primarily divide into three classes. First come such as are absolutely meaningless, the word *blictri* being the classical example of nouns of this type. Second come substantives so defined as to connote contradictory attributes; as examples we may cite the phrases "honest thief" and "chaste prostitute."

[17] P. 43.

Such a substantive (unless taken in a non-natural sense) cannot be the name of an object of thought, but it ought not to be regarded as entirely meaningless and might conveniently be spoken of as the name of a chimera. Third come names of objects, substantives having meanings attached to them which do not involve the connoting of contradictory attributes. A conception of an object is, of course, itself an object, and likewise the name of an object is an object which by an exceptional use of language (a *suppositio materialis*) may denote itself instead of the object or objects it ordinarily denotes.[18] There is no such object as a chimera (as we propose to use that word), for to say a substantive is the name of a chimera is to assert that there can be no corresponding object of thought. But the name of a chimera, and even a meaningless substantive, and in general every word or phrase, is an object. One of the first steps in philosophy is to distinguish carefully between a name, a conception corresponding to a name, and an object denoted by a name; the last being in exceptional cases the name itself or another name. In framing a philosophical nomenclature one should do all in his power to enforce these distinctions, but it is not too

[18] The distinction between the primary use of a word or phrase and its use in a *suppositio materialis* is quite ancient, dating back to the medieval logicians. Mr. Russell however seems to be serenely unaware of any such distinction, and by ignoring it he is enabled to put forth an argument that must be characterized as only worthy of a schoolboy. He says (p. 48): "It is plain, to begin with, that the concept which occurs in the verbal noun is the very same as that which occurs as verb. This results from the previous argument, that every constituent of every proposition must, on penalty of self-contradiction, be capable of being made a logical subject. If we say '*kills* does not mean the same as *to kill*,' we have already made *kills* a subject, and we cannot say that the concept expressed by the word *kills* cannot be made a subject." Here Mr. Russell, in attempting to show that a verb such as "kills" and a verbal noun such as "to kill" have precisely the same meaning, seeks support in the absurd contention that both words are alike subjicible. The facts are, of course, that "kills" or any other word or phrase can be made a subject by a *suppositio materialis,* but when we consider the *primary* uses of words and phrases there is a clear demarcation between those which can and those which cannot be made logical subjects. In this use, which is the only one relevant to Mr. Russells' argument, "kills" is not, while "to kill" is, subjicible. And the sentence "*kills* does not mean the same as *to kill,*" though making a statement about the primary uses of "kills" and "to kill," is not itself a proposition in which these are given their primary uses, but one in which "kills" and "to kill" each occur in a *suppositio materialis.*

much to say that Mr. Russell, in his nomenclature, gives the impression of being at great pains to obscure them. His use of "term" is a case in point. A philosopher need pay little heed to the colloquial uses of this word, and he may likewise disregard certain authorized applications of it that are not very common. But in logic and mathematics there are well-established uses of "term" which a writer on the philosophy of mathematics should take into account before assigning a new significance to it. In logic a term is simply a class name; thus "man" or "men" is a term, while the phrases "every man" and "some men" are names but not terms. In mathematics a term is sometimes a name; thus a term of a polynomial is an algebraic expression. The word is however also used in connection with series, and here a term is not a name at all, but is a number or other quantity.[19] These meanings of "term" are entirely disregarded in the nomenclature put forward by Mr. Russell. He adopts none of them, but deliberately tears the word from its accepted logical and mathematical uses. Although there is already available for his purpose a suitable name "individual" (or "individual object") he wantonly disregards the law of lexicological economy, and makes "term" synonymous with "individual" and two other words besides. And this is done in a work dealing with logic and mathematics, where there is urgent need for the use of "term" in its proper senses! Mr. Russell makes no attempt to furnish a new word to be used in designating logical terms, nor does he provide us with a specific title for the mathematical terms of either series or polynomials. In all three cases he leaves us in the lurch, and in mathematics, no less than in logic, the adoption of Mr. Russell's terminology would put a great hindrance in the way of precision of thought and speech.

[19] We must warn the reader that we are here not using "quantity" in a sense of which Mr. Russell would approve. He would not call a number a quantity.

We hold, then, that philosophical inquiry has its interests best subserved by using *individual* or *individual object* where Mr. Russell prescribes the use of "term." To use "unit" in Mr. Russell's sense cannot be deemed correct, for an individual is not taken as unit in every discussion in which it enters. The fourth synonym, "entity," is as objectionable as "term," just for the very reason that its use implies every object of thought to have existence "in some sense." Or rather, as Mr. Russell himself would put it, not "existence," but "being," for he draws a distinction between the two. Speaking of any pair of "terms," A and B, he says they "need not exist, but must, like any thing which can be mentioned, have Being. The distinction of Being and existence is important, and is well illustrated by the process of counting. What can be counted must be something and must certainly *be,* though it need by no means be possessed of the further privilege of existence. Thus what we demand of the terms of our collection is merely that each should be an entity."[20] In emblazoning the word "Being" with an initial capital letter, Mr. Russell would seem to imply that this word marked something of great importance. Yet in truth the basis for ascribing Being to "anything which can be mentioned" and to "whatever can be counted" is just one insignificant fact: the fact that the name of anything which can be mentioned is subjicible, forming a sentence when some set of words beginning with "is" or "is not" is put after it; in other words, by conjoining some predicate to this name the object of thought it represents can always be said to be or not to be something. And it surely cannot be held that the mere subjicibility of a name confers, on the object this represents, a mysterious something called "Being."[21] Are

[20] P. 71.

[21] The question of existential import of propositions (which however Mr. Russell does not appear at all to have in view) brings up other considerations. It can be plausibly contended that when we enunciate an affirmative propo-

we to take purely verbal reasons for our ground, and say that the sea-serpent being inexistent, and the author of the *Principles of Mathematics* being existent, both must be put under a common category of objects possessing Being? No advantages attend this course, and there is no reason why we should adopt it. It would substantially be arbitrarily agreeing to use "has Being" in the sense of "can be mentioned." And clearly the statement that "Such and such an object of thought has Being" is utterly futile if it merely means, as it does in Mr. Russell's phraseology, that the object of thought can have a name given to it. If such modes of speech as "has being," "is an entity," etc., are to convey anything worth putting into words we must have a dichotomization of objects of thought into those that have being and those that do not—into entities and non-entities. But apart from the dichotomization marked by the names "existent objects" and "inexistent objects" there seems to be none in connection with which the words "being" and "entity" can be conveniently used. So these words are best made synonymous respectively with "existence" and "existing object." Any object of thought can, of course, be spoken of as having existence in a hypothetical universe, and in talking of that universe we can call the object an entity; but we must not on this account call such an object an entity when there is under discussion any other universe (either the actual one or a new hypothetical universe). The right to be designated as an entity is given, and given only, by existence in the universe under discussion. To take any other view would result in depriving us of a very useful

sition with a certain name as subject we thereby, in certain cases, assert the object to which this name refers to have existence. Indeed the doctrine that affirmative propositions have existential implications, but negative not, is a part of the Aristotelian logic—at least the canons of that logic will not all hold unless this be true of A, E, I and O. Evidently however reservations must be made for such propositions as "So and so is a nonentity," it being denied that the affirmative form here gives an implication of existence, or else propositions of this character must be ruled out of philosophical language entirely.

word, for we could then no longer mark out by the aid of "entity" the existent objects of a specified universe from the non-existent.

In classifying objects as entities and non-entities one has most frequently the actual universe in view. In any event however it is from a cross classification that we obtain the two classes: individual objects and collections of objects. Whether an object is to be regarded as individual or as a collection depends to some extent upon the exigencies of the moment. Thus it is often convenient to take a collection of soldiers collectively as an individual regiment or as an individual army. An object again may be a perception (under which head we would put all states of consciousness) or an object whose existence is founded on a perception or a more or less complicated set of perceptions. Under this second head come sounds (and hence spoken syllables, words and sentences), sights (and hence written letters, words and sentences), etc. Here also come all bodies and the attributes of bodies as well as various other objects which are greater removes from their elemental perceptions. Conceptions are a species of perceptions. A name may denote a conception, and moreover a name which does not do this may have a conception corresponding to it and to the object it denotes. With each type of names that do not denote conceptions there arises a question as to what sort of conceptions, if any, corresponds to names of this type, and the debate over such a problem has not seldom given rise to a discussion classical in philosophical history. For instance, one school of philosophers would say that when a man speaks of any particular triangle he has in his mind a conception corresponding to this triangle and to the individual name that denotes it, and that moreover he may have a conception of a triangle that is "neither oblique nor rectangle, neither equilateral, equicrural nor scalenon, but all and none of these at once"

corresponding to the general class name *triangle*. Another school, while admitting the existence of conceptions corresponding to proper individual names (i. e., names put forward as each denoting a single object), even when such a name is the name of a class (a collection which is here taken collectively as an individual), would deny altogether that a general name could have any conception corresponding to it; one might have, they would say, a conception of a particular triangle, or of a collection comprising many triangles, but none of triangles in general. Mr. Russell does not stop to take up any such question, but plunges at once into a classification of individual objects ("terms") which is replete with confusion. "Among terms," says Mr. Russell, "it is possible to distinguish two kinds, which I shall call respectively *things* and *concepts*. The former are the terms indicated by proper names, the latter those indicated by all other words. Here proper names are to be understood in a somewhat wider sense than is usual, and things also are to be understood as embracing all particular points and instants, and many other entities not commonly called things. Among concepts, again, two kinds at least must be distinguished, namely those indicated by adjectives and those indicated by verbs. The former will often be called predicates or class-concepts; the latter are always or almost always relations."[22] Here the name "adjective" would seem to be applied to *words* of certain species, but just two pages previous we find Mr. Russell states: "What we wish to obtain is a classification, not of words, but of ideas; I shall therefore call adjectives or predicates all notions which are capable of being such, even in a form in which grammar would call them substantives." Any comment on the confusion here exhibited between language and what language conveys would be superfluous. Such confusion is

[22] P. 44.

habitual with the author of the *Principles of Mathematics*; indeed we might say it is the corner-stone of his system of "philosophy."

Searching for more light upon what Mr. Russell means by "concept," two remarkable discoveries will be made by the patient reader: first, that concepts *occur in propositions*; second, that concepts *denote*. "A concept *denotes* when, if it occurs in a proposition, the proposition is not about the concept, but about a term connected in a certain peculiar way with the concept. If I say 'I met a man' the proposition is not about *a man*: this is a concept which does not walk the streets, but lives in the shadowy limbo of the logic-books. What I met was a thing, not a concept, an actual man."[23] The old philosophical view was that a concept was a state of consciousness. Mr. Russell's concepts, it seems, are to be found, not in the mind, but in the "shadowy limbo of the logic-books." In the logic-books themselves we can find nothing but words. Since Mr. Russell declares that "a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words,"[24] and as he puts *word* and *concept* in antithesis, saying that we "employ words as symbols for concepts,"[25] he apparently does not regard a concept as a word or set of words. Had we been able to find any "shadowy limbo" in the neighborhood of the logic-books on our shelves we should have bravely ventured into it, and gone through those regions where that dreadful monstrosity, Symbolic Logic, "has its lair,"[26] seeking to encounter Mr. Russell's mysterious concepts, even at the peril of finding them to be Jabberwocks. But the shadowy limbo is not to be found, and filled with disap-

[23] P. 53. On this page *a man*, we are told, is a "concept," but on page 54 Mr. Russell informs us that "*A man*, we shall find, is neither a concept nor a term"! Here he not merely contradicts his statement on the previous page, but puts "concept" and "term" in an antithesis which is assuredly not in accord with his view that concepts are a species of terms.

[24] P. 47. [25] P. 53. [26] P. 66.

pointment we are compelled to abandon the hope of thus ascertaining what Mr. Russell means. At all events a concept, whatever that may be, can "occur in a proposition."

The significance Mr. Russell gives to "proposition" goes hand in hand with his peculiar use of "term." He would base a formal definition of the former word upon the motion of implication: "It may be observed that, although implication is indefinable, *proposition* can be defined. Every proposition implies itself, and whatever is not a proposition implies nothing. Hence to say '*p* is a proposition' is equivalent to saying '*p* implies *p*'; and this equivalence may be used to define propositions."[27] A less technical presentation of Mr. Russell's view is his statement that "A proposition, we may say, is anything that is true or that is false."[28] He also tells us that "propositions are commonly regarded as (1) true or false, (2) mental. Holding as I do that what is true or false is not generally mental, I require a name for the true or false as such, and this name can scarcely be other than *proposition*."[29] Most students of philosophy will be inclined to demur at the remark that propositions are commonly regarded as mental. Rather, they will say, the common practice is to regard a proposition as a written or spoken sentence, corresponding to which there may be a mental *judgment*. Here however this question is of no great importance; we are concerned, not with the practice that is prevalent, but with Mr. Russell's alone. As we have seen, Mr. Russell appears to accept "linguistic propositions" only as an unimportant species of the genus propositions, and when he speaks of propositions not mental he does not seem to have these specifically in view. Given a written or spoken sentence concerning Socrates, Mr. Russell would not ordinarily direct his attention to this sentence in considering what he

[27] P. 66 [28] P. 15. [29] P. ix.

calls the proposition. In the "proposition" which Mr. Russell would ordinarily have in view there occurs, Mr. Russell would say, Socrates himself, and not merely the name of Socrates.[30] *True proposition,* then, in Mr. Russell's sense would seem to mean *fact.* But while one can concede that John Smith may be intelligibly said to occur in the fact or state of affairs described by the sentence, "John Smith is chopping wood," when this sentence is true, it is not easy to see how John Smith (or Socrates) himself can "occur" in anything at all corresponding to a false statement about him. If Mr. Bertrand Russell, by some esoteric method, has discovered that Socrates himself, not merely the name or conception of Socrates, actually occurs in *something* corresponding to each false statement about Socrates, then it is most unfortunate that Mr. Russell's investigations have not enabled him to inform his readers precisely what this something is. And we must record our opinion that Mr. Russell's use of "proposition" is well fitted for only one purpose: the promoting a confusion between a sentence and the fact or fancy with which that sentence is conversant, just as his use of "term" is eminently adapted to create confusion between a name and what that name denotes.

Let us now return to the doctrines expounded by Mr. Russell in matters really mathematical. Part III of his work is devoted to "Quantity." The sense in which he uses this word is somewhat peculiar. He does not take it in what is probably its most suitable use, viz., its use as a generic name applicable at once to concrete numbers, to such denominate numbers as lengths, weights, areas, vol-

[30] We have already cited Mr. Russell's statement that "a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words." On the same page he refers to "the confusion," which he says arises from "the notion that *words* occur in propositions, which in turn is due to the notion that propositions are essentially mental and are to be identified with cognitions." And on page 45 he tells us that "Socrates is a thing, because Socrates can never occur otherwise than as a term in a proposition."

umes, etc., and to the natural numbers and the other abstract quantities analogous to these (e. g., those of such values as 0, ½, π, −1, $+\sqrt{-1}$, etc.). On the contrary he puts "number" and "quantity" in antithesis, and speaks of "applying numbers to quantities."[31] He also puts an antithesis between "quantity" and "magnitude." "An actual foot rule," he tells us, "is a quantity; its length is a magnitude. . . . when two quantities are equal they have the *same* magnitude."[32] Here we see again in evidence the blunder of thinking that like attributes are identical; that, for example, two bodies equal in length have the same length. Indeed it is upon this blunder that Mr. Russell founds his criterion for distinguishing between quantities and magnitudes. "A quantity is anything which is capable of quantitative equality to something else." "A magnitude. . . . is to be defined as anything which is greater or less than something else." And there is, Mr. Russell holds, "reason to think. . . . that what can be greater or less than some term can never be equal to any term whatever, and *vice versa.*"

With number attributes, as we have seen, Mr. Russell does not even put the question whether what are colloquially called the same attribute common to different groups ought not more properly to be regarded as like but different attributes. With "magnitudes" however he does give some attention to what he calls "the relative theory" which denies that there is any such thing as "a magnitude shared by equal quantities."[33] His "refutation" of this theory is worthy of note. He lays down eight indemonstrable axioms which he says the theory obliges us to assume, one of these

[31] P. 157. He tells us, page 159, that "In fixing the meaning of such a term as *quantity* or *magnitude*, one is faced with the difficulty that however one may define the word, one must appear to depart from usage." This is one of the many instances in which he slips away from the meaning he nominally gives "term," and uses the word in the customary logical sense.

[32] P. 159. On page 165 he tells us that "all magnitudes are simple concepts."

[33] P. 162.

axioms affirming that, whenever A is a quantity, "A being given, there is always a B [likewise a quantity] which may be identical with A, such that A = B," or, as he also puts it, "If A be a quantity, then A = A." He tells us that "These axioms, it will be observed, lead to the conclusion that, in any proposition asserting equality, excess, or defect, an equal quantity may be substituted anywhere without affecting the truth or falsity of the proposition. Further the proposition A = A is an essential part of the theory. Now the first of these facts strongly suggests that what is relevant in quantitative propositions is not the actual quantity, but some property which it shares with other equal quantities. And this suggestion is almost demonstrated by the second fact A = A. For it may be laid down that the only unanalyzable symmetrical and transitive relation which a term can have to itself is identity, if this indeed be a relation."[34] And Mr. Russell goes on to show—to his own satisfaction, though to us his argument is by no means cogent—that the admission of equality as a symmetrical transitive relation which an individual object ("term") can have to itself leads to a theory of magnitude which is not the "relative theory" at all, but the "absolute theory" that he himself upholds. We shall not here consider whether the first "fact" is really a fact—whether so broad a statement as to the substitution of [symbols of] equals can properly be made in an algebra developed on a philosophical basis. For in any event we are quite unable to see that this "first fact" can give the suggestion which Mr. Russell accredits to it. And as for the "second fact," on which Mr. Russell relies for the definite refutation of the "relative theory": the necessity in that theory for a quantity to be equal to itself, this may really be essential to the theory elaborated by Mr. Russell's fancy as a straw man for him to overthrow. But it cer-

[34] P. 163.

tainly is not the case that an algebra sanctioned by a true philosophy will ever affirm A to equal A. This affirmation is seen not to express a fact at all on making the proper distinction between equality and identity, and recognizing that the mathematical character $=$ does not always concern true equality, but has sometimes reference to identity. It is absurd to say that because mathematicians write $A = A$ and read this "A equals A," a quantity must necessarily be equal to itself. A philosopher ought not to take the customs of mathematical symbolism as foundation for his doctrines.

Another statement of Mr. Russell's bearing on the import of $=$ is of interest enough to cite here. "Among magnitudes, equality...has an absolutely rigid and unique meaning: it applies only to quantities, and means that they have the *same* magnitude.... Among numbers....there is no such thing as equality. There is identity, and there is the relation which is usually expressed by the sign of equality, as in the equation $2 \times 3 = 6$. This relation had puzzled those who endeavored to philosophize about arithmetic, until it was explained by Professor Peano. When one term of the equation is a single number, while the other is an expression composed of two or more numbers, the equation expresses the fact that the class defined by the expression contains only one term, which is the single number on the other side of the equation."[35] As a matter of fact, the explanation which Mr. Russell ascribes to Professor Peano is wholly erroneous. The numeral 6 does not denote a single number: it denotes many numbers; all

[35] P. 341. With his habitual confusion of thought and speech, Mr. Russell here leaves us in doubt as to whether the equation is a sentence written symbolically with numeral expressions as the two members, or whether it is something that the sentence merely serves to express, its "terms" being what the members of the sentence represent. Indeed his words might be fairly construed to imply that the equation is made up of a number (be it noted that he does not say numeral or symbol of a number) on one side and a mere expression on the other, the expression in some strange way being "composed of two or more numbers"!

those comprised in the value class of the sixes. Likewise the expression 2×3 denotes a multitude of numbers, all of which are sixes, but not all sixes are denoted by this expression. For not every six is the result of an operation of multiplying a three by a two, and only the numbers resulting from such operations are entitled to be designated by the expression 2×3. And $2 \times 3 = 6$ means: Every product of a two into a three is equal to or identical with every six (i. e., is equal to every six save that with which it is identical). So here $=$ has reference to both equality and identity.

From quantities[36] we may proceed to the consideration of variables. The conception of a variable is a most important one in mathematics, and can be traced back to the attempts made by Archimedes and other Greek mathematicians at the rectification and the quadrature of circles. In this work it was essential to conceive of a variable and the limit which the latter approached, though it was not until many centuries later that "variable" or any equivalent name was heard of in mathematics. The use of the word does not date back to much before the time of Newton, and the earlier definitions all ascribed to a variable the character of a quantity, quantities being classified as constants and variables. Definitions of this type are by no means obsolete. Thus in a work on differential and integral calculus, published not very many years ago under the auspices of Professor Peano, we are told that "In the questions considered there may appear quantities to which determined and fixed values are supposed to be attributed, and these are called constants, and other quantities supposed to be able to assume diverse values, and these are called variables."[37] An examination however of the vari-

[36] The word "quantity" we shall hereafter use, not in Mr. Russell's sense, but in what we esteem to be the proper one.

[37] *Calcolo Differenziale e principii di Calcolo Integrale* by A. Genocchi, "Publicato con aggiunto dal Dr. Giuseppe Peano."

ables that mathematics takes within its scope shows that such a definition is quite unsuitable. The question as to whether a variable can properly be termed a quantity is closely related to the problem of sameness and similarity of attributes, possessed by different groups or different individuals, which arises in connection with the natural numbers and other quantitative attributes (e. g., lengths, weights, volumes, areas, etc.). We decided in the foregoing pages that two groups alike in number ought not to be said to have the *same* number, but to have different though *equal* numbers, and that two bodies alike in length ought not to be said to have the *same* length, but to have different though *equal* lengths. And so with weights, volumes, areas, etc.

Suppose now that there is only a single body in view, and that a change takes place in reference to one or more attributes. Suppose, for instance, that a bar of metal is heated, and its length changes from 1000 centimeters to 1020 centimeters, while its color changes from black to red. Are the facts well expressed by saying the bar possesses the "same" length and the "same" color as before? If the old definition of variable is to be taken as criterion, we must answer in the affirmative as regards the length at least. For the length of the bar during the change is what would be called a *variable* in mathematics, and the definition in question tells us a variable is a quantity—that is, *one* quantity. And should we take a like ground in reference to color attributes we must say that the manifold colors which appear throughout the change are a color, that is, *one* color. If however we refuse to be bound by an ancient definition, and prefer the only course sanctioned by a sound philosophy—that of so using language as to mark the distinctions to which our senses testify, then we must recognize that such a use of "same," implying identity where there is diversity, is even more repugnant than

its application in the sense of equal. And we see that when an object initially possesses a certain attribute and undergoes a change with respect to this attribute, we must then regard as a distinct and separate attribute each stage of the process of change.[38] Such a set of attributes may together constitute a variable, and under this description come most of the variables of physical science. We may have a body with variable velocity—a type of variable which gave frequent opportunity for the exercise of Newton's genius—a body of gas with variable temperature, under variable pressure and with variable volume. We cannot say that any of these variables are quantities unless we wish our language to be an impediment, instead of an aid, to exact thinking.

Even stronger do we find our case when we turn to geometry, and consider the type of variable that first appears in history. Archimedes, in striving to rectify a circle, inscribed in it first an equilateral triangle, then a regular hexagon, then a regular dodecahedron, etc. The perimeters of the inscribed polygons here constituted a variable whose limit was the circumference of the circle, while the areas of the polygons constituted a variable whose limit was the area of the circle. With what show of reason can we say that the perimeter of a triangle is one and the same quantity—one and the same perimeter—as the longer perimeter of a hexagon? How are we justified in saying that the area of a triangle and the area of a much larger ninety-six-sided figure are the same quantity? And yet we must make these assertions if we acknowledge that a

[38] The stages recognized by our naked organs of sense are comparatively few. The various instruments at our disposal for assisting the senses enable us greatly to increase the number of stages that can be deliminated, but this number always remains finite. To assume that a succession of suitable instruments would, if they were at hand, enable us to keep up a continual increase *ad infinitum* in the number of stages detected, and hence to ascribe innumerable stages to the change where we really do not observe so many (which is in effect what is usually done in scientific work) is to adopt a hypothesis, legitimate enough as such, but not to be taken as anything more than a hypothesis.

variable is a quantity. Nor is it only with circles that such difficulties arise. The quadrature (or rectification) of any curvilinear figure by the methods of the modern integral calculus presents similar difficulties. We have a set of rectilinear figures, no two alike, the areas of which (or the lengths of certain lines on which) constitute a variable the limit of which is sought. Surely these considerations are overwhelming against the doctrine which would ascribe to a variable the character of a quantity, and there is no recourse save to abandon that doctrine completely.

A variable, then, is not a quantity, but is constituted by a set of quantities, and the quantities in the set may be natural numbers or other quantities of the same sort, or be concrete quantities, or be such denominate quantities as lengths, areas, etc., It remains for us to ascertain the essential characteristics of such a set, for sets of quantities which do not constitute variables also come under consideration in mathematics. Now the characteristic investigations in which variables appear in mathematics are undoubtedly those where limits are concerned. And it is to be noted that a limit (which is a quantity that may or may not itself belong to the variable) pertains, not to any one quantity in the variable taken separately, but either to the whole set of quantities or to a part of that set so extensive as to comprise innumerable quantities. When it is said that the variable x approaches the limit l, it is not meant that any quantity of x approaches l; such a statement would be utterly meaningless. If, for instance, we have a variable composed of unique representatives of the values 1, 1½, 1¾, 1⅞, etc. arranged in the order here given, then this variable approaches as limit a quantity of value two (a limit which is in this case not in the variable) but it would be absurd to speak of the 1 or of the 1½, or of any other quantity of the variable as approaching a limit.

Again a variable composed of unique representatives of all real abstract values from 1 to 3 arranged in order of value first approaches a limit of value two (which is here in the variable), then attains it, and finally recedes from it, but we cannot say that any separate quantity of the variable approaches or recedes from this or any other limit. A variable may approach a limit throughout its whole extent or throughout only a portion of its range, but in no event can a quantity of the variable be said to approach a limit. And we may lay down that the purpose of taking a set of quantities together as constituting a variable is to investigate matters that concern the mutual relations between quantities of the set, this being the case whether a limit is or is not concerned. Here then is found the essential characteristic of the set of quantities which constitutes a variable: the purpose for which it is formed. There is a sharp contrast in this respect between the sets of quantities comprised in a variable and other sets of quantities which appear in mathematics. Take, for example, the value classes. When we form a value class, and give it a class name, as when we group together all the twos, and provide the class name "two," our purpose is to investigate questions and state propositions concerning every two taken separately or concerning each of some of the twos. When we say $2 + 3 = 5$, we assert that every two plus any three is equal to or identical with every five, an assertion wholly unlike that made in "The variable x approaches the limit l."

On first thought it might appear appropriate to specify, as another distinguishing feature of variables, the taking of diverse values. It seems only natural to say that there must be variation of value in a variable. But to take etymology as decisive in laying down a definition is the method of the pseudo-scientiest, not that of the really scientific investigator. On inquiry into the matter we indeed find that variation is not always at hand. Mathematicians have

even coined a specific name, *tratto d'invariabilita* in Italian, *Invariabilitätszug* in German, to designate a portion of a variable (here the dependent variable of a functional relation) in which there is conformity to a single value, and not variation. And moreover there are variables with which variation of value is everywhere lacking. For in analytical geometry the mathematician finds it convenient to regard the abscissas of a line parallel to the y axis (or the ordinates of a line parallel to the x axis) as constituting a variable, though obviously in such a variable all the quantities are of a single value.

While in many mathematical works the ancient definition of variable still lingers on, a new type of definition is beginning to appear in some of the more modern treatises. The innovation is however unobtrusively set down without any word to show that a conception of a variable is being put forth which is different from that embodied in the old definitions. These new definitions describe a variable as a symbol. Thus in that most authoritative of mathematical works, the *Encyklopädie der mathematischen Wissenschaften,* Professor Pringsheim, in his article on the *Grundlagen der allgemeinen Funktionenlehre,* lays down that "By a real variable is to be understood a *symbol,* usually one of the letters of the alphabet, to which is assigned successive different number-values (for example, all possible between two fixed number-values, all rational, all integral)."[39] The only way in which this definition has reference to real variables as opposed to variables in general, is by the use of "number-values" (i. e., real values[40]) instead of simply "values." So in Pringsheim's view a variable would seem to be essentially a symbol to which there are assigned in succession various different

[39] Vol. II, Part I, p. 8.

[40] "Numerical value" would of course not be a correct translation of *Zahlenwerte,* though this is sometimes so translated when used in just the sense taken for it by Pringsheim.

values. To those who are satisfied to accept manipulation of symbols as the Ultima Thule of mathematics, defining a variable as a symbol will doubtless be satisfactory. Such a course produces quite a different impression however upon those who would no more take as the subject matter of mathematics the words and symbols it makes use of than they would rest satisfied with a science of botany which studied such "words" as *coniferae and cruciferae,* and did not investigate the objects these words stand for. Variables in mathematics are not symbols any more than numbers or other quantities are symbols. Variables and quantities are both things represented by symbols. Pringsheim's definition and others of the same type have really reference, not to variables, but to the names or symbols of variables. So construed they seem to mean that when and only when a symbol takes successively a number of different values (that is to say, denotes in succession quantities of these respective values) then that symbol is the symbol of a variable. How very remote this view is from the actual facts in the case is sufficiently obvious from the investigations into the constitution of variables that have been made in the foregoing paragraphs.[41] The most characteristic use of the name or symbol of a variable is, we have seen, not in propositions about the quantities of the variable taken individually, and in the characteristic use (e. g., when we say x approaches a limit) the symbol *does not take any value at all.* It is true that mathematics also sanctions using the symbol of a variable in quite another way—in the equations dealing with functional relations—but this latter use is by no means distinctive of the sym-

[41] One or two mathematicians cursorily define a variable as an *aggregate* of quantities, but go no further into the question of the characteristics of a variable, and say nothing of the distinctive way in which the name of a variable is used. Such a definition is not, on its face, so untenable as the quantity definitions or the symbol definitions, but we cannot regard it as entirely satisfactory. For a discussion of this matter however we must refer the reader to our forthcoming work: *Fundamental Conceptions of Modern Mathematics.*

bols of variables. And before we consider it we shall proceed to see what doctrine Mr. Russell holds as to the nature of variables.

In Mr. Russell's system variables are of prime importance. The reader is introduced to them at the very outset of the *Principles of Mathematics.* "Mathematical propositions," Mr. Russell says, "are not only characterized by the fact that they assert implications, but also by the fact that they contain *variables.*"[42] In his use of the term "variable" Mr. Russell is not content to abide by the ordinary mathematical customs, but seeks to give a much broader field to its application. "It is customary in mathematics to regard our variables as restricted to certain classes: in arithmetic, for instance, they are supposed to stand for numbers. But this only means that *if* they stand for numbers, they satisfy some formula, i. e., the hypothesis that they are numbers implies the formula. This, then, is what is really asserted, and in this proposition it is no longer necessary that our variables should be numbers: the implication holds equally when they are not so. Thus, for example, the proposition 'x and y are numbers implies $(x + y)^2 = x^2 + 2xy + y^2$' holds equally if for x and y we substitute Socrates and Plato. [It is necessary to suppose arithmetical addition and multiplication defined (as may easily be done) so that the above formula remains significant when x and y are not numbers] : both hypothesis and consequence in this case, will be false, but the implication will still be true. Thus in every proposition of pure mathematics, when fully stated, the variables have an absolutely unrestricted field: any conceivable entity may be substituted for any one of the variables without impairing the truth of our proposition."[43]

[42] P. 5.

[43] P. 6. It should be remembered that in Mr. Russell's view "propositions" are not sentences, and that in a proposition "occur," not names, but the things and concepts designated by names. Hence to substitute the entity Socrates

Even in ordinary mathematics Mr. Russell gives variables a rôle much more important than that usually assigned to them: "The variable is, from the formal standpoint, *the* characteristic notion of mathematics. Moreover it is *the* method of stating general theorems.... That the variable characterizes mathematics will be generally admitted, though it is not generally perceived to be present in elementary arithmetic. Elementary arithmetic, as taught to children, is characterized by the fact that the *numbers* occurring in it are constants; the answer to any schoolboy's sum is obtainable without propositions concerning *any* number. But the fact that this is the case can only be proved by the help of propositions about any number, and thus we are led from schoolboy's arithmetic to the arithmetic which uses letters for numbers and proves general theorems.... Now the difference consists simply in this, that our numbers have now become variables instead of being constants. We now prove theorems concerning n, not concerning 3 or 4 or any other particular number...Originally, no doubt, the variable was conceived dynamically, as something which changes with the lapse of time, or, as is said, as something which successively assumed all values of a certain class. This view cannot be too soon dismissed. If a theorem is proved concerning n, it must not be supposed that n is a kind of arithmetical Proteus, which is 1 on Sundays and 2 on Mondays, and so on. Nor must it be supposed that n simultaneously assumes all values. If n stands for any integer, we cannot say that n is 1, nor yet that it is 2, nor yet that it is any other particular number. In fact, n just denotes *any* number, and this is sometimes quite distinct from each and all of the numbers. It is not true that 1 is any number, though it is true that whatever holds of any number holds of 1. The variable,

for x in a proposition is not, it would seem, to substitute the *name Socrates* for the symbol x, but to actually take Socrates out of his grave, and put him, in some incomprehensible way, in the place of the object represented by x!

in short, requires the indefinable notion of *any*."[44] This "indefinable notion of *any*" is put by Mr. Russell in sharp contrast to the notions of "all" and of "every." "*All a*'s denotes a numerical conjunction.... The concept *all a*'s is a perfectly definite single concept, which denotes the terms of *a* taken altogether.... *Every a*, on the contrary, though it still denotes all the *a*'s, denotes them in a different way, i. e., severally instead of collectively. *Any a* denotes only one *a*, but it is wholly irrelevant which it denotes, and what is said will be equally true whichever it may be. Moreover *any a* denotes a variable *a*, that is, whatever particular *a* we may fasten upon, it is quite certain that *any a* does not denote that one; and yet of that one any proposition is true which is true of any *a*."[45]

Mr. Russell further tells us: "We may distinguish what may be called the true or formal variable from the restricted variable. *Any term* is a concept denoting the true variable; if *u* be a class not containing all terms, *any u* denotes a restricted variable. The terms included in the object denoted by the defining concept of a variable are called the *values* of the variable: thus every value of a variable is constant."[46]

Finally we are told what a variable is: "Thus *x* the variable is what is denoted by *any term*."[47] "Thus *x* is in some sense the object denoted by *any term*, yet this can hardly be strictly maintained, for different variables may occur in a proposition, yet the object denoted by *any term*, one would suppose, is unique. This however elicits a new point in the theory of denoting, namely that *any term* does not denote, properly speaking, an assemblage of terms, but denotes one term, only not one particular definite term. Thus *any term* denotes different terms in different places."[48] "The notion of the variable....is exceedingly complicated. The *x* is not simply *any* term, but any term with a certain

[44] P. 90. [45] P. 58. [46] P. 91. [47] P. 6. [48] P. 94.

individuality; for if not, any two variables would be indistinguishable."[49]

Mr. Russell's extension of the word "variable" to cases where what are in question are not quantities is quite in line with his other innovations in terminology. At all of these we are inclined to demur. We shall not however dwell upon such matters here, but will restrict ourselves to inquiring whether Mr. Russell's doctrine, in its application to cases where quantities do happen to be concerned, conforms to the true theory of the variables met with in actual mathematical work. Now it is quite evident, on examining Mr. Russell's remarks, that his idea of a variable has not arisen from a systematic consideration of the variables of mathematics. Mr. Russell's conception of a variable is essentially an etymological one, and what he takes to be variables are not variables at all. We have shown that, with a variable, variation of a value is neither sufficient nor necessary. Mr. Russell however thinks that taking different values is the very essence of a variable. In this view, "If x and y are numbers, $(x + y)^2 = x^2 + 2xy + y^2$" is a typical case of a proposition involving the symbols of variables. And he holds that whenever we formulate a proposition concerning n instead of merely concerning 1 or 2 or 3 or some other particular number then we are dealing with variables. The facts are, to begin with, that a proposition about what Mr. Russell calls a particular number, for example 3 in the equation $3 + 2 = 5$, is about a whole species of numbers—the value class of the threes, or rather it is about every number in this species taken individually. And when we go to a general theorem involving such a symbol as n we merely ascend to a genus comprising several (usually innumerable) value classes. When n is used in what Mr. Russell describes as a proposition about any number, it is a class name denoting every number of every

[49] P. 107.

value class whatsoever, and the propositions to which Mr. Russell refers as "about *any* number" may with perfect propriety be said to be about each and every number. Take, for instance, the equation $(n+1)(n-1)=n^2-1$. This affirms: *With every number whatsoever the sum of this number plus any one, multiplied into the difference of the same number minus any one, is equal to or identical with the remainder obtained by subtracting any one from the square of the number in question.* It is true that we can read the equation as a proposition in "any number" if we prefer to do so. We can render it: *The sum of any number plus any one, multiplied into the difference of the same number minus any one, is equal to or identical with the remainder obtained by subtracting any one from the square of the number in question.* But there is no diversity of import between this and the rendition first given. The "any" version and the "every" version have precisely the same meaning. The difference is not a philosophical one, but resides merely in the syntactical construction of the one sentence requiring "any" (or "each," which could just as well be used) while the syntax of the other permits the use of "every." In taking the curious view that a general theorem of arithmetic involving such a symbol as n is characterised by being "about *any* number," and in implying, as he does by the opposition in which he puts "any" and "every," that it is not about *every* number, Mr. Russell is propounding a doctrine which is utterly untenable. Between "any" and the n (or other symbol) of a general theorem there is not really the connection imagined by Mr. Russell, and the vague subtleties which fill several pages devoted to "any" in the *Principles of Mathematics* are beside the point. What connection there is lies between the symbol and the whole set of universal syncategorems,[50]

[50] We use "syncategorem" as the title of those words and phrases which are conjoined to class names to indicate the logical quantity. Sometimes "syn-

every, each, any, etc., and does not pertain to any one of the set to the exclusion of the rest. As regards the class to which such a symbol refers the theorem is universal—is a proposition in which the class name of that class is distributed—and if it is to be put into words some universal syncategorem must be conjoined to the class name to indicate the distribution. In the purely symbolic mode of expression sanctioned by mathematics there are however no characters representing syncategorems; these are always taken for granted, mathematics dealing so largely with generalities that whenever a formula occurs the tacit assumption is made that this must be universal with respect to every symbol involved. These symbols are class names to which syncategorems must be conjoined when we pass from formulas to equipollent sentences in which we retain the symbols but none of the signs or other characters of the originals. And thus the *n,* in a proposition about every number ("about *any* number," as Mr. Russell would put it), is a synonym of "number"; it is a class name for numbers of all species. Like other class names, it by itself denotes every member of its class, while when conjoined to a syncategorem every member is or is not denoted, according to whether the syncategorem is or is not universal. In the general theorems of mathematics, since a universal syncategorem is understood though not expressed, every member of the class, that is, every quantity (or every quantity of the type in question, e. g., every number) is denoted, and hence the symbol does simultaneously assume all values (or all values of the type), Mr. Russell to the contrary notwithstanding.

The confusion of such a symbol as Mr. Russell's *n*—a general symbol for quantities of many values—with the symbol of a variable is not peculiar to Mr. Russell, and

categorematic word or phrase" is used in a broader sense in which it applies also to certain words that appear as *parts* of class names, e. g., of, in, at.

seems to be largely due to a one-sided view of the symbol of a variable. There are in mathematics important propositions in which symbols of variables are used in a way which enables them to be construed as here denoting the quantities of variables, just as the class name belonging to a class of quantities denotes the quantities of that class. These propositions are the equations concerned with functional relations between two or more variables. Thus y and x being two variables, there might be a functional relation between these that would in mathematical symbolism be indicated by $y=x^2$, the significance of this equation being that a functional relation[51] subsists under which every quantity in the variable y which has a corresponding quantity in the variable x is equal to or identical with the square of the corresponding quantity in x. And if we choose, we may use the shorter phrase "every y"[52] instead of "every quantity in y," and say "every x" instead of "every quantity in x." With this mode of reading, if one considers only such equations and neglects the more characteristic use of the name or symbol of a variable, it is not surprising that he should confound the names of variables with ordinary class names. There is really an analogy between such a proposition as "Every y having an x corresponding to it is equal to or identical with the square of

[51] We hold that a functional relation between two variables is established whenever there is, first, correspondence between quantities of the two variables and, second, likeness in order of corresponding quantities. It is not essential that *every* quantity in either variable should have a corresponding quantity in the other. For a discussion of this matter we refer to the work already mentioned.

[52] Here y does take values. But the way in which it does so is not by having successive different values assigned to it as Pringsheim wishes us to believe. It does not assume the values represented in the variable *successively*; it takes all of them *simultaneously*. There may be consecution connected with a variable, but it is not a consecution of the times in which a symbol takes different values; it is a consecution in the order of arrangement of the quantities constituting the variable. For the quantities of the variable have commonly an arrangement in order—an arrangement which with some variables is immutable but with others is not. An arrangement under which the quantities are consecutive may, as we have said, occur, but other arrangements are common, and some variables are not even capable of having their quantities arranged in a sequence. So it is certainly not a happy thought to bring succession into the definition of a variable.

the corresponding x," and a proposition in which occurs the class name of a class of quantities formed solely for the study of properties which the quantities of that class possess as individuals. Be it noted however that the closest analogy is not found with those general propositions of mathematics which Mr. Russell takes as typical propositions involving the symbols of variables. Thus consider $a + b = b + a$ which, when put forward as enunciating the commutative law of addition, is a fair example of these general propositions. This should be read: "Every a when any b of the same sort[53] is added to it gives a sum equal to or identical with the sum of the same b as before plus the same a as before." Here we have what might be called an equation of sameness; in reading it we are obliged to specify that the second a taken is the same as the first a, and that the second b is also the same as the first b. Otherwise the proposition would not be true; for a and b are both general class symbols each of which takes all values and denotes every quantity that enters mathematics; and hence if we did not put the specification as to sameness we should be asserting that every sum of two quantities is equal to or identical with every sum of two quantities, whether the quantities entering the second sum were the same as those entering the first or not. Now the equations concerning functional relations do not always carry a stipulation as to sameness; this may be at hand, as in $y = x^3 - x^2$ where the same x must be squared in forming the subtrahend as was cubed to form the minuend, or it may not be needed at all as in $y = x^2$. On the other hand these equations always carry a stipulation totally lacking with $a + b = b + a$: namely a stipulation as to correspondence. For the equality and identity alternative in equations concerning a functional relation

[53] It is necessary to specify that the b taken must be of the same sort as the a. For if it is of a different sort the addition cannot be performed. Thus we cannot add a length to a weight, or add either to a natural number.

between a y-variable and an x-variable holds only of a y and of its *corresponding* x. Stipulations of correspondence, though not at hand in such equations as $a + b = b + a$, can however be found where ordinary class symbols are alone involved. The special equations that arise in connection with certain problems bear such stipulations. Take for example the system of equations $x + y - 10 = 0$, $2xy - 9x - 11y + 54 = 0$, where x and y are symbols of unknown quantities—*quesitive symbols* as we might call them. These equations are satisfied by $x = 7$, $y = 3$, and by $x = 4$, $y = 6$. Here x denotes quantities of two different values, and so does y. We cannot read $x + y = 10$ as "Every sum of an x plus a y is equal to or identical with every ten"; this is not true, for $7 + 6$ is not 10, nor is $4 + 3$. We must say "Every sum of an x plus the corresponding y," and must likewise read the other equation of the system as an equation of correspondence. In quite another way, correspondence is in evidence with the general equations each of which covers a host of special cases. Such an equation is the general equation of the second degree with one quesitive symbol: $ax^2 + bx + c = 0$, which on solution gives $x = (-b \pm \sqrt{b^2 - 4ac})/2a$. This solution, as well as the original equation, is understood to carry stipulations as to correspondence; the equations do not have reference to each set of values that can be formed from the values taken separately by the four literal symbols—if they did the first of the equations would assert such absurdities as $3 = 0$; for x, a, b and c all take the value 1. The equations are understood to hold only for each set of corresponding values; in other words the equality or identity alternative is asserted only of an x, an a, a b and a c which correspond. Mathematicians who deal with the theory of equations not infrequently use the name "variable" in connection with the x of such equations as these, but decline to speak of variables where a, b and c are concerned, thus

taking the ground that the symbol of a variable must be quesitive. They ought however to know that in what is usually regarded as the natural habitat of variables—the differential and integral calculus—it is by no means a requisite that the quantities of a variable be unknown. If concerned in a correspondence or taking diverse values or both together be accepted as criterion, then clearly a, b and c in the general equation as well as x ought to be designated as symbols of variables, and so likewise ought x and y in the system of special equations. But the real criterion has nothing to do with variation of value, and nothing to do with correspondence (which does not come into consideration in the constitution of a variable, but only appears on the stage when a functional relation is established between several variables); and none of these symbols are symbols of variables. A set of quantities constituting a variable is characterized, not by the values of its components or by any correspondence into which they enter, but by the purpose for which the set was formed, or, if we wish to be perfectly precise, the purpose which it serves. Take x in the general equation of the second degree: the purpose in view in grouping together the quantities called x's is not to investigate their mutual relations, but merely to facilitate finding the value of the x of each instance that may come to hand as a special case of this general equation, and thus to avoid the trouble of formulating and solving a new equation for each separate case. The x's (and the a's and the b's and the c's) may be made to constitute a variable by carrying out an investigation which does bear on their mutual relations, but when initially grouped together they assuredly do not pertain to a variable at all.

In considering Mr. Russell's doctrine that the ordinary class symbols of mathematics are names of variables, we have. let the reader remember, made due allowance for

the fact that he did not well choose the examples he gave of the former, but failed to put forward those most analogous to the latter as used in equations dealing with functional relations. We have gone out of our way to bring to light the cases where the greatest analogy is to be found, thus giving the most favorable aspect to Mr. Russell's contention. And the result of our investigations is assuredly not to sanction the identification of symbols of variables and ordinary class symbols used only to denote the separate members of classes of quantities. Between the symbols of these two types there is, we see, a difference of character which makes Mr. Russell's doctrine utterly untenable. This doctrine, ignoring as it does the really characteristic use of names of variables, must be deemed quite unworthy of a philosopher. What extenuation can be urged in Mr. Russell's behalf? The only one we can conceive as being put forward is the plea that Mr. Russell had a perfect right to use "variable" in a sense peculiar to himself, and to give it, in the discipline he miscalls "mathematics," an application quite different from that which the word has in the mathematics of the mathematicians. Now we willingly concede that deviations from the established meanings of important scientific terms, frequently as Mr. Russell indulges in them, ought not to be looked upon with intolerance. We would not reproach him merely for this: such offenses against current usage are in themselves only venial sins for a philosopher. But, though they are of small moment alone, it is quite a different matter when they have as concomitant a confusion of thought which pervades a whole philosophy and seem to play no small part in bringing about this confusion. And that is precisely what we find with the innovations fathered by Mr. Russell, who, moreover, while admitting that in many cases he apparently departs widely from common usage, does not always seem aware of the extent of his

transgressions.[54] Notably with the very word "variable" Mr. Russell is to all appearances serenely unconscious of his flagrant contravention of the terminology sanctioned by scientific use. His application of the name, so far as he uses it in connection with the symbols occurring in ordinary mathematical formulas, he takes as a matter of course, though in fact nothing is more unwarranted than his procedure here. After his initial error we can hardly be surprised at his extension of this application, under which he designates as names of variables the phrases obtained by conjoining "any" to even those general class names that do not pertain to what is ordinarily called mathematics. The extension is a natural result of Mr. Russell's assigning to mathematics the broad field of symbolic logic, and indeed, the misconception of variable is taken as basis for the formal definition of "Pure Mathematics" promulgated in the *Principles*. "Pure Mathematics," Mr. Russell informs us, "is the class of all propositions of the form 'p implies q,' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants."[55] Elsewhere he tells us, "The variable is, from the formal standpoint, *the* characteristic notion of Mathematics."[56] And he actually says that his definition of Pure Mathematics "professes to be, not an arbitrary decision

[54] Of all the words in a language "not" is one of the few whose meanings are firmly established by a usage uniformly alike in scientific and in colloquial discourse. Will it be believed that Mr. Russell is found making a use of this word utterly opposed to the meaning which has heretofore been universally assigned to it? Speaking of relations (p. 25) he notes that "with some relations, such as identity, diversity, equality, inequality, the converse is the same as the original relation: such relations are called *symmetrical*." And he then proceeds to say: "When the converse is incompatible with the original relation, as in such cases as greater and less, I call the relation *asymmetrical*; in intermediate cases, *not-symmetrical*." Needless to say, in the sense in which "not" had been universally used up to the time Mr. Russell began to write, the asymmetrical and the "not-symmetrical" relations would alike be designated as *not symmetrical*. Yet there is nothing in his remarks to show that Mr. Russell was conscious of how greatly "not" is distorted from its accepted meaning in the nomenclature he devised.

[55] P. 3.

[56] P. 90.

to use a common word in an uncommon significance, but rather a precise analysis of the ideas which, more or less unconsciously, are implied in the ordinary employment of the term."[57] The use of the word "variable" in Mr. Russell's definition may perhaps so impose upon some of his unthinking readers as to make them swallow this contention, but if that word be elided, and a less misleading synonym be put in its place, then Mr. Russell's analysis of the ideas implied in the ordinary employment of the term "Pure Mathematics" will even to the most casual reader appear anything but precise. For what Mr. Russell really means by a variable here is either a *general class name, distributed by the conjunction of a universal categorem or otherwise,* or else the object represented by the distributed class name—the hazy nature of his doctrines makes it difficult to tell which. And it is perfectly ludicrous to say that Pure Mathematics, in the ordinary acceptance of this term, is characterized by such distributed class names or by what they stand for.

We have seen then that in what are perhaps the two questions most fundamental in mathematical philosophy—the doctrine of numbers and the doctrine of variables—Mr. Russell's failure is complete and utter. His delineation of numbers is inadequate, and the definition he puts forward is inadmissible. What he regards as variables are not the variables of mathematics, nor has he penetrated into the nature of what he wrongly takes to be variables. He is to all appearances unaware that "any individual" (which in his view denotes the true variable) and "any *u*" (which he regards as denoting a restricted variable) are nothing more than distributed general class names. He assumes that "any" has always an office radically different from that of "every," and puts forth many would-be subtleties in a laborious attempt to show just how the two syn-

[57] P. 3.

categorems differ.[58] Starting from an erroneous assumption, he succeeds in bringing obscurity over a matter that is on its face perfectly clear, and is after all brought to admit that for "the variable," whose nature is, in Mr. Russell's opinion, inextricably bound up with the question of "any," a satisfactory theory is yet to be found.[59] Rightly or wrongly, after going over Mr. Russell's theories, one feels inclined to ascribe the errors he commits very largely to the curious metaphysics which here and there makes itself manifest. An inkling of Mr. Russell's metaphysics may be gained from what he propounds concerning propositions and concerning objects and terms, things and concepts. In dealing with the latter he puts forward a classification—substantially a doctrine of categories—which appears to be quite original and is likewise preposterous. Had Mr. Russell deferred his work in the field of mathematics until he had elaborated as foundation a sane and coherent doctrine on these points and others of pure philosophy, he might not have so rapidly written a ponderous volume, but assuredly he would have been more likely to have produced a work of real benefit to science. As it is, one can only regret that the author of the *Principles of Mathematics* should have expended so much labor in erecting an imposing edifice upon foundations that are thoroughly unsound.

ROBERT P. RICHARDSON,
PHILADELPHIA, PA. EDWARD H. LANDIS.

[58] In fact, as we have said before, opposition between "any" and "every" is often quite lacking, and, an "any" sentence can in such a case be changed into an "every" sentence with but slight modification. However opposition does sometimes occur, but not in the way Mr. Russell would have us think. Opposition occurs with the potential mood; thus, "Any horse may win the race" is truly opposed to "Every horse may win the race." When we pass to the affirmative indicative and compare, for example, "Every horse is a mare" and "Any horse is a mare," we are no longer able to find opposition; the latter sentence is not of a usual form, but if we grant it any meaning we will naturally accept it as equipollent to the former. With the negative indicative there is however opposition, "Not every horse is a mare" and "Not any horse is a mare" being opposed,. (We need hardly say that we do not use "opposed" in the sense of "incompatible.")

[59] P. 5.

THE DEFINITION OF NUMBER.

"SCIENCE," writes Gaston Milhaud,[1] "in enouncing its ever-increasing series of truths, obviously supplies—whether one reflect upon it or not—the most powerful argument against scepticism. And in this respect mathematics plays a special rôle by reason of the evidence which clothes all its propositions and by reason of the complete satisfaction which its demonstrations give to our thirst for comprehension. There, at least, is a domain where thought in search of clarity, of evidence, and of light, exercises itself in an ideal fashion. Everywhere else, discussion is founded on the right to proclaim as certain an enounced truth, and accord upon the value and legitimacy of each insight comes but slowly: in mathematics this is not so. If, for the choice of axioms, we give ourselves voluntarily to philosophical investigations whose conclusions vary, at bottom there is no one ready to abandon the postulates of ancient geometry, and the question was not even proposed by the Greeks. As to demonstrations, it seems impossible that two minds, however different they may be—granting their disposal, at need, of obvious misunderstandings—will not speedily agree upon the rigor of the reasoning, and consequently upon the rigor of the conclusions. And whether one is aware of it or not, the habit of such a movement of thought creates in us a naive confidence in the puissance of our under-

[1] *Les Philosophes-géomètres de la Grèce*, 1900; pp. 2-3.

standing,—so that it would be a miracle if the philosophical geometer did not somewhere testify to it, did not sometimes under the most penetrating conceptions bear along a disconcerting dogmatism."

We may take this statement, I think, as a fair representation at once of the fascination and the dangers which beset mathematical reasonings. There is no field of human thought, I imagine, which yields so paradoxical a feeling of freedom and of constraint as does mathematics: the freedom springing from the twofold consciousness, first, of our having chosen the postulates from which we proceed, and second, of the endlessness of the possible elaborations of our reasonings; the constraint arising from our sense of the undeniableness, and therefore the necessity, of mathematical demonstrations,—i. e., from their freedom from contradiction. Thus from mathematics we derive the satisfaction which our instinct for law and order always yields in finding itself fulfilled, without at the same time sacrificing our self-gratifying conviction of the importance of the human factor in the operations performed. In the study of physical nature there is always a certain abasement of humanity, due to the passive attitude of scientific observation, accompanied by a feeling of outer and brute constraint; but the mathematician, with an even greater assurance of the necessity of his results, bears with him also a lively consciousness of the significance of his own activity in bringing about these results, and so attains, as it were, a kind of Zeus-like supremacy to the fated ends which, while they bind him, are yet his own enactment.

But is not this doubly reason for caution against mathematical dogmatism? and especially that form of it which rests its denial of our more ordinary intuitions, not upon its eventual translatability, but upon its untranslatability into the forms of our common human experience? Doubtless truth is difficult and obscure; but dare we concede that

it is so ineffably obscure as to transcend the discourse of life? Of course I am speaking of the modern science of logistic.[2]

I.

What is the meaning of number? and in what sense are the hairs of our heads and the other phenomena of nature numbered? This is the question.

The old-fashioned view of number found its essence to lie in *discontinuity* coupled with a notion of *series.* "Number is discontinuous," says Clerk-Maxwell;[3] "we pass from one number to the next *per saltum.*" The perception of the discontinuity was regarded as empirical and intuitive. In the language of Aristotle, "We perceive number by the negation of continuity, and also by the special senses, for each sensation is a unity."[4] The perception of the series was usually accredited to the act of counting, though this was often also somewhat confusedly regarded as an act of adding. If I speak of this view in a past tense, it is only because of its long history; not that it is dead.

In the thinking of such men as Hobbes and Locke this conception eventuates in an out-and-out nominalism. "Number," quoth Hobbes,[5] "is exposed either by the exposition of points or of the names of number, *one, two, three, etc.*; and those points must not be contiguous, so as that they cannot be distinguished by notes, but they must be so placed that they may be *discerned* one from another; for from this it is that number is called *discrete quantity,* whereas all quantity which is designed by motion is called *continual quantity.* But that number may be exposed by the names of number it is necessary that they be recited

[2] Which is, by the way, a somewhat unhappy name; for with the Greeks "arithmetic" was the more, "logistic" the less theoretic science.

[3] *Encyc. Brit.*, 9th ed., III, 37.

[4] *De Anima,* 425a, 5.

[5] *Concerning Body,* XII, 5.

by heart and in order, as one, two, three, etc.; for by saying one, one, one, and so forward, we know not what number we are at beyond two or three; which also appear to us in this manner not as number, but as figure."

It is always worth while citing Locke in connections of this kind, not because of the analytical value of his expositions, which is usually slight, but because he gives, with a dogmatic perspicuousness that leaves nothing to be desired, the first reflections of ordinary common sense. He says:[6] "By the repeating the idea of an unit and joining it to another unit, we make thereof one collective idea marked by the name two: and whosoever can do this, and proceed on, still adding one more to the last collective idea which he had of any number and gave a name to it, may count, or have ideas for several collections of units distinguished from one another, as far as he hath a series of names for following numbers, and a memory to retain that series with their several names; all numeration being but still the adding of one unit more, and giving to the whole together, as comprehended in one idea, a new or distinct name or sign, whereby to know it from those before and after, and distinguish it from every smaller or greater multitude of units. So that he that can add one to one, and so to two, and so go on with his tale, taking still with him the distinct names belonging to every progression; and so again, by subtracting an unit from each collection, retreat and lessen them; is capable of all the ideas of number within the compass of his language, or for which he hath names, though perhaps not of more."

In this account it is obvious that Locke presupposes: (*a*) the notion of *unity*, which, indeed, he has just previously stated to have "no shadow of variety or composition in it"; (*b*) the notion of a *collection*—his "collective idea"; (*c*) the notion of *serial order*; (*d*) the notion of *quantity*

[6] *Essay*, II, xvi, 5.

—greater and less; (*e*) the notion of a mathematical *operation*—addition, subtraction. Thus the main elements in the concept he is describing are assumed; at the same time there may be a seasoning of hard-headedness in his stout nominalism. For him numbers are names: "Without names or marks we can hardly make use of numbers in reckoning, especially where the combination is made up of any great multitude of units, which, put together without a name or mark to distinguish that precise collection, will hardly be kept from being a heap in confusion." One of the primary issues in the modern discussion of the nature of number is just whether supersensible (or super-intuitible) mathematical ideas do not resolve into mere nomenclature and the science itself into a kind of transcendental logomachy.

That the Lockean type of nominalism is by no means extinct is evidenced by the definition of number offered in the *Encyclopaedia Britannica*:[7] "Suppose we fix on a certain sequence of names 'one,' 'two,' 'three,'. . . ., or symbols such as 1, 2, 3,. . . .; this sequence being always the same. If we take a set of concrete objects, and name them in succession 'one,' 'two,' 'three,'. . . . naming each once and once only, we shall not get beyond a certain name, e. g., 'six.' Then, in saying that the number of objects is six, what we mean is that the name of the last object named is six. We therefore only require a definite law for the formation of the successive names or symbols. The symbols 1, 2,. . . .9, 10,. . . ., for instance, are formed according to a definite law; and in giving 253 as the *number* of a set of objects we mean that if we attach to them the symbols 1, 2, 3,. . . . in succession, according to this law, the symbol attached to the last object will be 253. If we say that this act of attaching a symbol has been performed 253 times, then 253 is an *abstract* (or *pure*) *number*.

[7] 11th ed., article "Arithmetic."

Underlying this definition," continues the writer, "is a certain assumption, viz., that if we take the objects in a different order, the last symbol attached will still be 253. This, in an elementary treatment of the subject, must be regarded as axiomatic; but it is really a simple case of mathematical induction."

The presupposition of discontinuity and of serial order is as obvious in this last as in the two previously given accounts of the number concept. We set out with our known power of observing differences and naming things —perceptual discrimination and apperceptional unification; but by the time we have accomplished the office of Adam and are taking our earned rest, we discover that the names we have given are vicariously indifferent to the things of "first intention," and in addition that they have won for themselves a wholly novel and stringent interdependence, —the smoke of our experience has transformed itself into a hugely articulate Jinni, and, as by a miracle, number is manifest! Aristotle says,[8] "In general what exists in the essence of number, besides quantity, is quality; for the essence of each number is what it is when taken once, 6 being not what it is when taken twice or thrice, but what it is once, that is, 6." It is very apparent that a succession of qualitative discriminations will not in itself yield quantity; and without an understanding of quantity how can number be defined?

II.

The ideal of the logisticians (though I speak with misgivings) is at once the infallibility and the universal applicability of their reasonings. They would create for us a rational universe entirely freed from the taint of empiricism, mathematical in its certainties, but hypermathematical in its significance,—in short, they would

[8] *Metaph.*, 1020*b*.

achieve what Spinoza so greatly attempted. Because of the annoying miasmas which beset the earth-born speech of men, they would substitute therefor a kind of Esperanto of the soul (*anima intellectiva*) modeled after the discarnate and purified symbolism of mathematics. Clearly the approach to this consummacy of the intellect should be through the concept of number.

First of all, this concept must be relieved of all traces of Lockean empiricism. The simple notion, prevalent among the ordinary, that the idea of number is in some fashion derived from the act of counting is one of which we must be eased. For what is meant by counting? "To this question we usually get only some irrelevant psychological answer, as, that counting consists in successive acts of attention. In order to count 10, I suppose that ten acts of attention are required: certainly a most useful definition of the number 10!"[9] The point is well taken, and we can see that it applies conclusively to the whole British tradition, from Hobbes onward. "We must not, therefore, bring in counting where the definition of numbers is in question."

To be sure, this judgment has not prevailed in the new school *ab initio*. Dedekind states that from examination of what takes place in counting an aggregate of things, we are brought to consider the mind's powers (*a*) of relating things to things, and (*b*) of letting a thing correspond with, or represent, a thing; and that upon these powers as a foundation the whole science of number must be based.[10] *Relation* and *equivalence* are thus fundamental ideas—or, perhaps, operations—which get their meaning from counting, and give its meaning to number; but it may be that the counting here meant is of that purely noetic variety which includes "denumeration" of the in-

[9] Russell, *Principles of Mathematics*, p. 114.
[10] *Was sind und was sollen die Zahlen?*

finite along with "enumeration" of the finite, and which, putatively, owes no dependence to our commoner experience.

But if not counting, then neither is mathematical induction the key to the meaning of number; for mathematical induction, with its dual stress upon *next-to-next* and *recurrence*, is no more than the act of counting transubstantiated by that unity-in-variety which is the root of all perception. "We may define finite numbers as those that can be reached by mathematical induction, starting from 0 and increasing by 1 at each step,"[11] but such a definition does not apply to the vastly greater realm of transfinite numbers,—and it would be obvious waste to devote thought to a definition applicable only to the "little corner," as Poincaré calls it, "where the finite numbers hide themselves."

By what device, then, are we to pry into the mystery of number? What idea — which the mutations of the Wheel of Time have brought back to us freed from the contaminations of a too mortal birth—will give us its eluctant essence? The answer is familiar: A finite cardinal number is a class of equivalent classes; an infinite cardinal number is a class of classes a part of which is equivalent to the whole. It is the idea of *class* which is to resolve for us the riddle of reasoning.

Readily enough our imaginations seize the suggestion. The older, empirical conception of number as somehow directly derived from the act of counting, in reason as in history, is replaced by one in which counting and all other operations flow from an initial insight into a group situation. The point of regard has been reversed, and in place of seeing a perceptual situation built up out of moments, we see the moments emerge from the situation; logical priorism disenthrones empiricism, deduction precedes in-

[11] Russell, *op. cit.*, p. 123.

duction—and, indeed, not unnaturally absorbs the latter, for any induction which may lay a claim to reason is but deduction disguised.[12]

But this might flow from a mere distinction of temperament;[13] for we have long been accustomed to Urania and Pandemos in reason as in love. The matter which calls for a nicer determination is the relation of this term *class* to its content. What does it mean?

It is difficult to be precise in the analysis of terms which are customarily defined only by a set of properties couched in the form of postulates. What one arrives at is a word (*flatus vocis*) with a variety of meanings, but meanings eviscerated of that heart of reality which we feel to be present in our more current, if less critical, living speech. Indeed, all that saves this rarified discourse from the emptiness of nominalism is the requirement of consistency as between the postulates; their freedom from mutual contradiction is their sole claim to a single and central meaning. This (if I understand it) is the only principle of definition recognized in logistic.

What, then, are the properties of a "class"? Clearly, I think, the prime requisite is that it shall constitute a *limit*. I do not mean a limit which conveys a sense of a beyond (if that can be avoided), but a limit which clarifies our sense of the within,—such a limit as, for example, is represented by the cardinal number of the class of finite numbers, or again, such a limit as we ordinarily intend by the word "universe." Without this conscious limitation, which, because we feel it to be a voluntary intellectual retrenchment, a kind of rein upon the imagination, we personify

[12] *Ibid.*, pp. 11*n.*, 441.

[13] This, apparently, is Poincaré's notion of his own divergence from Russell. "M. Russell me dira," he says, "qu'il ne s'agit pas de psychologie, mais de logique et d'épistémologie; et moi, je serai conduit à répondre qu'il n'y a pas de logique et d'épistémologie indépendantes de la psychologie; et cette profession de foi clora probablement la discussion parce qu'elle mettra en évidence une irrémédiable divergence de vues."—*Dernières pensées*, p. 139.

as a "self-limitation," no conception of class could be operant.

Dedekind's solution of the problem of continuity quite consciously rests upon the assumption of *limits,* or limiting values; and what is distinctive of the notion of a "cut" (*Schnitt*) appears to be just that it determines a limit which, so to speak, does not overleap itself, and which consequently gives the base for a self-contained system of values. Every "cut" is, in a sense, a zero, having the particular property that any variable magnitude which approaches the limit loses itself in a value indistinguishable from zero.[14] This, I take it, is also the essential meaning of the *Nul class*—the class of things to which no entity in the (given) universe corresponds; it is essentially a boundary which, because it is empty, cannot be used as a turn or start into continued reasonings.

At least we should suppose that 0-limits could not be so used, but by a kind of transcendental induction just this is attempted. The cardinal number of all finite numbers, which is, of course, infinite, becomes the first transfinite cardinal; and the ordinal ω (ω symbolizes a *progression* modeled on the natural suite $1, 2, 3, \ldots . n, n + 1, \ldots .$, and so may be regarded as the generalization or law of the process of ordering sequentially) becomes the first transfinite ordinal. By applying the conception of a transfinite ordinal to transfinite cardinals, it becomes possible to conceive of, and perhaps create, an infinite series of the latter —transfinites of the order α being followed by those of the order β, and so on. The whole process is reminiscent of Spinoza's assumption of possible infinite attributes, other than thought and extension, of the divine substance, though it seems to want the restraint which left Spinoza content to suggest the possibility, and pass in his philosophizing to the attributive planes with which human experience famil-

[14] *Stetigkeit und irrationale Zahlen,* IV.

iarizes us. By means of such interplays of conception—infinite limiting finite, transfinite limiting infinite—it becomes possible to create whole hierarchies of classes and types, each conclusively including what is below it and conclusively ignoring what is above it. The process is interesting and in its way fruitful, but it is difficult to see how it could be possible except for that self-imposition of limits which distinguish grade from grade and type from type, and it is difficult to see in the imposition any other necessity than the arbitrary will of the thinker. The limits set are limits assumed, and assumed with something of the stark inexplicableness of a primitive tabu—unless we concede that the whole process is a conscious fiction, whose analogue is our empirical concentration of immediate attention on immediate ends.

But besides this external principle of limitation, which makes definable a self-comprehending system, there is another principle of limitation, an internal one, which makes system itself comprehensible. This principle is represented by the idea of *structure* or *form,* without which mathematics and reason alike could not exist. The principle of external limitation might suffice to mark off for us an islet of chaos which we could choose to regard as the universe, but only the acknowledgment of internal limitations could convert this chaotic universe into a cosmos.

Now the relationships of ideas according to this principle of internal limitation assume two general forms: that of *part-to-part* and that of *part-to-whole.* It is obvious[15] that each of these is a relation of *order,* and it is also obvious that each is derivative of the idea of *unity* in the two fundamental senses of unity. For the relation of part-to-whole clearly rests upon the contrasting unities of the element, regarded as an undifferentiated item, and the thing, regarded as an assemblage of elements; and the

[15] "Obvious," not to logistic, but to our linguistic intuitions.

relation of part-to-part, while explicitly concerned only with the relation of item to item, clearly rests upon an implicit whole.[16] Unit and totality, atom and universe, are the two extremes, each of which assumes the mask of unity, and the fact that the atom may be resolved into a universe or the universe contracted into an atom by a simple act of speculative translation does not alter the essential character of these two moments of thought.

The relation of part-to-part would, in the world experientially familiar to us, involve the meaning "next-to-next," or contiguity of consecutive elements. This relation is what makes the experiential world finite and incomplete; it is, therefore, felt as a constraint of the pure reason, mathematical or other. But the logisticians have discovered an escape from this restriction, and like Spinoza have found their freedom *sub specie aeternitatis*. The instrument of emancipation is the notion of the "one-to-one correspondence"; it is through this that the infinite is resolved

[16] For the two types of unity, cf. Bergson, *Données immédiates,* pp. 58f. Of course the logisticians categorically deny that the idea of *class* involves that of "part-to-whole." "Socrates is a man" may mean (1) "Socrates possesses the qualities which mark a human being"—and this is the part-to-whole relationship—or (2) "Socrates is one among men,"—and this is the member-to-class relationship, expressed "x is an a" (symbolically, $x \epsilon a$), where x is a member and a a class. The distinction is true enough, and it is also true that only the part-to-whole relationship is "transitive," i. e., subject to syllogistic treatment. But is it not evident that the distinction is fundamentally the very distinction which a philosophy of number is called upon to explain? Reasoning *qualitatively,* i. e., where your terms are taken "by nature," we get judgments of type 1; reasoning *quantitatively,* i. e., with terms taken "in respect to number," we get those of type 2. In judgments of the member-to-class type number is assumed, not definitely as if counted, but indefinitely as if countable. That is, a plurality, which is a totality or aggregate of some sort (in so far as limited by the reasoning undertaken), is at least hypothetically "taken"; and such a plurality is what is meant by a "class" (except in those shadowy extremes where the class has only one member or none at all). But if there is a plurality or aggregate it must have the configuration of just this (whatever it may turn out to be) aggregate which is being dealt with—just the class in question. Such configuration (which we might call the quality of a quantity) is precisely a whole of which the member is a part,—at least, we use "whole" and "part" in this sense in common speech, and it is certainly significant that the logisticians, in denying that "class" has this meaning, are forced to proclaim the term undefinable except by its use—i. e., it is left in a state of empirical ambiguity. Cf. Russell, *Principles,* Chaps. II, VI; also, Burali-Forti and Padoa in Vol. III of *Bibliothèque du Congrès international de Philosophie.*

into cosmos. The idea is centrally that of the reciprocal uniformity of two groups (classes), such that for every element of the one there is in the other, one, and only one, corresponding element. Two groups or classes so related are said to have the same number, and the infinite is simply a group in which the whole is related to a part of itself in this manner.[17]

Now the notion of a one-to-one correspondence is clearly metempirical. In real life, we cannot make things correspond absolutely except in absolute identification, i. e., in loss of plurality; all other relationships involve some kind of contiguity. Even when we set five fingers against five fingers, what we have empirically is not a one-to-one correspondence of two groups, but right thumb to left thumb, right index to left, and so on; and this holds throughout the empirical universe. The idea of number is, as it were, interposed between the severally adjacent digits, or perhaps I had better say that the groups of five are groups of five because they both (speaking with Plato) "participate" in τὰ μαθηματικά. The correspondence lies between an empirical group which is always finite and incomplete and a metempirical system of numbers (supermundane, if not divine) representing the class of all possible classes. If Spinoza's divine substance, within which all attributes inhere, were to become articulate it would be represented, I conceive, by just such transcendental numbers.

But we are not to think of these numbers as severally interdependent. Their reality rests upon no idea of succession. We must think of decads, duads, monads, triads, tetrads, etc., not of one....two....three....etc. The *order* of the numbers in their own transcendent realm is something superposed upon their cardinal realities,—this

[17] The usual illustration is that of the one-to-one correspondence of all the integers with all the even integers, or of the points on a straight line with the points on a plane.

time by a set of relations which concerns them *inter se.* *Less than, greater than, equal to,* or again *higher* and *lower power,* or again *betweenness* (or "mediacy," since the notion of "between" is significant only when coupled with the idea of transition), are relations of the needed kind. Now each of these sets of conceptions is a variant of the *part-to-whole* relation, of contained and container. This is self-evident in the first-named group. "Less than" and "greater than" obviously rest upon the experiment of mensuration, of reduction to scale, and if the numbers themselves *are* the scale, nevertheless they get their steps or intervals, and hence their *order,* from the experiences whose comparisons they name. A scale may be regarded as made up from the sucessive remainders in a series of approximations, its fineness being determined by the extent to which the approximations are carried,—which, in last resort, must be a matter of industry or of organic structure, in either case empirical. The relation of equality is not so obviously derived from measure, for "equal" may signify not merely identity in step or scale, but also similiformity and equivalence. Nevertheless, when we consider that equivalence is no more than functional identity and that similiformity can be no less than this—that is, that each of these ideas is identity with a reservation—it would seem evident that here too we are dealing with a concept whose final meaning is derived from the part-to-whole relation.

In the case of "higher-lower power" and in the case of "betweenness" the same general relation—part-to-whole—is implicit. Both of these types of expression are derivatives of space-perception; they are geometric in first intention. But as principles of order they have to do not with a static but with a dynamic geometry. The notion of direction or sense is the primary one, but the direction exists not as the expression of an orientation but of a

progression; not a set of starting-points or markers but a set of journeys is connoted. Thus we have time as well as space involved in the empirical foundation of numerical order so conceived, the complete idea being the analogue of a movement from any assumed position in any designated direction, the movement being conceived as contained by its determinants. Of course, in the case of "betweenness" this movement may be ideal, and in that case we have merely a case of syllogistic transition, with the "between" represented by the middle term;—but this is simply intellectualizing our journey. Again, the concept of "betweenness" may give rise to right-left, symmetrical-asymmetrical orders; but here, too, we have only special complications of the familiar idea, for right-left are clearly but alternative journeys, a dilemma of roads one or the other of which our action must make real (hence defining the whole);[18] while symmetry and its opposite can hardly be conceived apart from measurement, for indeed the whole notion of proportion is dependent upon some kind of repetition (which again throws us back upon time and space for our analogues).

Thus the logistic conception of number, starting with the assumption of *class* as the essential numerical idea, proceeds in two directions. (*a*) Outwardly, it posits a limit or law within which must fall all the elements which make the class a class, capable of structure. And that this outward limitation is made in good faith as essential to the idea is sufficiently evidenced by the recognized possibility of a class including classes, of a class of classes, and finally of the class of all possible classes,—a veritable hierarchy of types of limitation. (*b*) Inwardly, there are posited two types of structural relation, which may be described as the principles of internal limitation. These are the re-

[18] So also "before-after." Past time is commonly thought as a *retreat* from the present, future time as an *advance*.

lation of part to part and part to whole. From the first is derived that freedom to make comparisons which makes possible—or, *is* the possibility of—the transcendental independence that distinguishes pure number. From the second flows the whole concept of order, and especially the notion of series or progression without which the idea of quantity (i. e., greater-less) could not be.

If we ask what concepts are fundamental in such a construction, three seem to stand predominant: class, element, relation. But the two first, class and element, are surely no other than the two meanings which we commonly ascribe to unity, while relation is quite as clearly *the* function (and therefore the meaning) of plurality. The *one* and the *many* are thus the fundamentals of number,—and already we seem to be within hailing distance of the Hellenic categories; subject and attribute, thing and quality, are recurrently proximate. Has the Wheel of Time indeed completed its circuit? and is philosophy to begin anew? Or were we perhaps right as to the distinction of temperaments, and is logistic but an exercise of the lovers of Uranian reason?

III.

At the beginning of my discussion I quoted from Gaston Milhaud a word of caution in regard to that dogmatism which issues from a too naive confidence in the powers of our understanding, especially when freed, as it is in mathematical logic, to consume its own intentions. I would repeat this caution, having in mind certain developments of this logic based upon the principles already examined.

These developments issue from that abstractive freedom which is the especial pitfall of the Uranian mind. When in a given situation a given form is discovered, the statement of this form is what we call the description of the situation, for it is only forms that we can state. But

a form so abstracted—and this is the law of our rational life—is invariably made the measure of new situations. The fact that it can never be applied to a new situation except with some more or less accommodating deformation is a fact which we customarily and conveniently neglect, or if we remember it, it is only for the sake of abstracting from the more comprehensive situation given by the group of deformations a new form of forms which shall serve in its turn for the first of a series of modifications of some super-form of forms, and so on;—i. e., *sf*...*df*...*df′*... *df″*...etc., as it becomes clogged by the impertinencies of fact, is clarified by being transmuted into FS...DF...DF′ ...DF″...etc., and this in turn by εἶδος ... ἐντελέχεια (α′...β′...γ′...), whence, we may presume, the Idea of Ideas breathlessly emerges as we pass above the sphere which bounds our empyrean. If this description be false to the process it has not yet been so demonstrated.

Now there are two modes in which this process is applied in the logistic analysis of number, corresponding to the two types of relation of a class to its limits which we have heretofore stated. These two modes might be described as the modes of external and internal transcendence of unity.

The first of these, the external transcendence, is effected by analogical reasoning the base of which is the so-called "natural" suite of numbers, the succession of positive integers 1, 2, 3,....n, $n+1$,... The number of such integers, which is infinite, is ω; but ω is more than this. It is also *the principle of description which is immanent in the natural numbers naturally arranged*; it is the principle of numerical order as evinced in one-to-one correspondences, and so is the key to the analysis of all denumerable groups. The postulates underlying descriptions of the type ω are (*a*) the postulates of *linear order,* and (*b*) postulates of *sequence*—Dedekind's for example. From the

combination of these two ideas issues the conception of a discrete series, though when we consider that the first of these is symbolized merely by the idea of inequality ($<, >$), i. e., by quantity, and the second by that of *limit,* i. e., by class, it does not appear that "discrete series" spells much more than "whole numbers." Nevertheless, as symbolized in ω, it becomes the beginning of a transfinite hierarchy of orders; for it is the principle (or, shall I say, the analogy) of the suite of finite numbers which sets in order the houses of the infinite,—there the last becomes first, Omega the prior of Alpha, and the unity of the finite integers is transcended by numbers α_ν reaching to the order 2ω, while beyond this we may suspect yet more transcendent orders of hyper-α's.

But this external transcendence of unity is complemented by an internal transcendence; there is not only a metempirical macrocosmos, but a metempirical microcosmos. This is shown forth when in the description of order the notion of sequence is replaced by that of *betweenness* or mediacy, which is to be conceived as a kind of eternal negation of next-to-nextness without loss of plurality. There are two kinds of numerical order exemplifying this internal transcendency. When a series is endlessly linear and yet endlessly median, i. e., when it has no beginning nor middle nor end but only and always a median term between any two terms, it is *dense*. When a series is limitedly linear but has no middle term, it is *continuous*. The endless fractioning of a difference in a process of approximation—as, for example, the endless interstitial fractions required to complete the suite of all rational numbers—is image of the dense series; the clogging of an interval by the sum of its own possibilities is the image of the continuous series—for example, the series $\geqq 0 \ldots\ldots \leqq 1$ is fulfilled by the aggregate of numbers rational and irrational there comprised. Series of each of these types are trans-

finite; but there is an important difference in structure between them, for only the dense series is denumerable (i. e., figurable by the progression of positive integers), while only the continuum is susceptible of ratio and of measure, for it alone has limit. Of course the dense series is only metempirically countable and the continuous series measurable only metempirically, so that to note that we seem to have here naught but a transcendentalizing of the Aristotelian πλῆθος and συνεχής, plurality and magnitude, is to suggest an empirical meaning for what is by definition beyond experience.

And yet is this suggestion without reason? The transcendentalities of logistic are accomplished in two directions, which might be termed the gross anatomy and the histological analysis of the number-cosmos; and yet, in order that the directions may be meaningful, must we not recognize some proximate and experiential greater-less which is the *here* from which we orient these directions? This seems clearly implied by the important rôle played by the conception of the suite of "natural" numbers, and again by that of the line, in the representation of order. Very likely it is true that finite numbers cannot be satisfactorily defined except in relation to transfinite classes, but can the transfinite be defined without first assuming the finite? As a matter of fact, the transfinite orders seem all to be got by a process of progressive abstraction and recombination of qualities assumed on the analogy of the natural numbers; it is as if, by a cunning complexity of mirrors, the natural suite were made to suffer indefinite distortions, variously deforming its native properties and translating them from plane to plane and from space to space in a succession of *saltus,* as many as one has patience for.[19]

[19] This right of saltation is clearly the foundation of the conception of transfinity. "In recent times there is arisen, in geometry and in particular in the theory of functions, a new type of conception of the infinite; according

The process is legitimate enough if we be not duped by its parlous illusions. That is, we must preserve our sanity (which is nothing less than our common-sense faith in our common-sense intuitions); and for this I can conceive no better rules than are implied in Aristotle's dicta (1) that when we speak with reason we must say something with a communicable meaning, and (2) that "third man" abstractions are wasted breaths.[20] The first of these is a pragmatic statement of the law of contradiction applied to discourse; the second is a caution against the tautology involved in the regress to infinity. If we adhere to the first we cannot shift our perspective (say, from finite to transfinite) without distortion of meaning, i. e., without altering our predication; if we adhere to the second we cannot make abstractions of abstractions without losing reality altogether.

Now it would seem that logistic fearlessly invites both of these perils. In the description of classes, for example, the forms of expression travesty the sense of language. For what can be the common-sense, linguistic meaning of a *Nul-class,* which must be described as that class which contains no element, or of which the universe (of discourse) furnishes no instance? Or again, by what right

to these new notions, in the study of an analytic function of a complex variable magnitude usage calls for the representation, in the plane which represents the complex variable, of a unique point situated in the infinite, that is to say, infinitely distant, but nevertheless determined, and for the examination of the manner in which this function comports itself in the neighborhood of this absolute point as in the neighborhood of any point whatever. It is seen then that the function in the neighborhood of the point infinitely remote acts precisely as it would act in that of every other point located in the finite, so that one is fully authorized in this case to represent the infinite as transported to a point altogether determined. When the infinite is presented in a form thus determined, I call it *infinite properly so-called.*"—G. Cantor, *Acta Mathematica,* 2, p. 382. Poincaré founds his conception of dimension upon the notion of the "cut" (*Dernières pensées,* p. 65; *La valeur de la science,* 97f), which, since it implies a new law in each new location, seems a more legitimate use of the right (or intuitive power, as Poincaré would make it) of overleaping boundaries ideally set. Plato's conception of the cosmos as made up of intervals and limits held together by proportion is not far from this (Timaeus, 35-36, 53-57; cf. Philebus, 14*c*-27*b*).

[20] *Metaph.,* 1006*a*.

of speech may we speak of the class *a* as the "class" which contains only *a*? In the first of these cases we are using the language of plurality about nothing, and in the second about one. And if we go a step further and speak of x' as the sole element of the class whose sole member is x, is this more than a vicious play upon the conception of part and whole?[21] Beyond this there is the x'' which is the sole content of the class whose sole element is x',—and we are fairly launched in the infinite tautologies of the "third man."

Formal refinements of analysis, when freed from the leadings of empirical need, may defeat the very end of analysis. Thought becomes not purified, but anemic; in a world of ideas where not only is language replaced by symbols, but these by symbols of symbols, all linguistically ineffable, it is small wonder that identities and the sense for the law of identity vanish away, so that no longer, in order to reason, do we need to speak significantly as Aristotle would require, nor indeed to speak at all. And with the disappearance of identities from this analytic attrition, it is but to be expected that there will emerge that "liberty of contradiction"[22] which solves infinity by denying sense and confounds truth with paradox. The ultimate reason of the world becomes a relation of relations which, if it could say anything, would say just that the world exists, catholically comprehensive of all contradictions, but which, since it is unutterable, is in so far inferior to the sacred monosyllable *Om*.

I am quite willing to agree that there is a sense in which this world is the best possible world, and indeed a sense in which it is the only possible world, and even a sense in which it is all possible worlds,—but when I have got so far I begin to suspect that I am being duped by my

[21] Perhaps I should mention that Burali-Forti *et al.* make distinction between "is an element of" and "is contained by."

[22] Cf. Poincaré, *Science et méthode*, pp. 195f.

own tongue and I deem it the modesty of reason to conserve my breath. Is there no like nonsensicality in the refutation of the axiom that the whole is greater than the part? And have we made "infinity" a more usable notion (I will not say in logistic, but in reason)[23] because we can juggle a part into a kind of equality with its whole by nominalizing our definitions? Common sense, we may be sure, will be slow to relinquish the intuitions upon which it acts; and will not our humaner reason itself, when it meets contradiction assuming the guise of infallible truth, begin to suspect that the ghosts of Duns and Occam are coruscating behind the scenes?

IV.

The attempts of the logisticians to define number by the unaided agilities of the reason are, in the end, little more satisfying than is the confident empiricism of Locke. No one can question their demonstrations, granted their premises; but no one, in the right mind of common sense, can grant the premises. It is incumbent upon us, then, to ask whether logistic has, after all, quite so efficiently scotched the particular theory whose downfall it proclaims,—I mean the intuitionism especially associated with the name of Kant.

"The pure form of all quantities for the outer sense,"

[23] I seem to discern among the logisticians themselves, when they are speaking the language of philosophy, a tendency to employ the idea of what Cantor terms "the infinite improperly so called" in place of that "properly-called" logistic infinite which could hardly be expected to convey an intelligible idea when severed from its nominalistic illustrations. And speaking of these illustrations, why should we stop with an infinite whose part equals its whole? Suppose we define Chaos as a nul-class ($X = 0$), and Cosmos as the class of all ordered classes, infinite in number ($K = \omega$). Then τὸ ὅλον, the Whole, (H), will be equal to a part of itself, a logistic infinity ($X + K = H$, or, $0 + \omega = \omega$). But suppose, in addition to infinite K, the Demiurge (since that is his business) determine other ordered classes (K'), as many as he may choose. K' will belong to K, as being ordered classes, but cannot add to the number of K which is infinite, nor to H with which K is already in a complete one-to-one correspondence. Then $K (= K + K') > H (= K + X)$, and the part is greater than the whole. Of course this is a play upon the idea of progression in time; perhaps none the less a fair image of the course of reason,—though "I hold it not honesty to have it thus set down."

says Kant,[24] "is space; the pure form of objects of sense in general is time. But the pure *schema of quantity,* as a concept of the understanding, is *number,* which is a representation conceptually combining the successive addition of unit to like unit. Thus number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, in that time itself is engendered in the apprehension of the intuition."

Thus for Kant "unity in the apprehension of a manifold" and "time," the empirical image of an *a priori* schema, are the fundamentals of the idea of number. We shall be not far wrong in identifying here the notions of unity, multiplicity, and serial order, which are primitive with Locke and are unevaded by the logisticians. But Kant puts these notions in a somewhat new light: they are no longer *bloss empirisch*, as with Locke, nor are they circuitously inferred from nominalistic definitions; rather, they come into being as elements of that synthetic activity which is the dominant mark of mind. Number is, in this sense, neither empirical nor quite metempirical. The categories of the understanding lie behind the numerical schema, but the schema itself is "only the phenomenon or sensible concept of an object in agreement with the category." Further, this schema—as indeed are schemata in general—is only the *a priori* determination of temporal intuitions, getting its content not through analytic but through esthetic transcendentalities. Indeed, one is tempted to say that Kant, like Plato, puts his mathematical realities in a kind of mid-realm participating at once in νοῦς and αἴσθησις.

The unique position of the number idea appears again in Kant's discussion of the formation of determinate numbers. Judgments of numerical relations, he says, are certainly *a priori* syntheses, but they are not, like the under-

[24] *Kritik der reinen Vernunft,* 182.

lying principles of geometry, universal in character. Accordingly, they are to be termed *number-formulas* (*Zahlformeln*), not axioms, and they are endless in number, i. e., as many as numbers themselves.[25] Kant conceives the formative judgments as synthetic apprehensions of aggregations of units. In their generation we may make use of sensible intuitions, as in computing by aid of the fingers, but the actual realization of a sum would be impossible apart from the *a priori* schema. "The arithmetical judgment is always synthetic, as may the better appear when we consider the larger numbers; for it is then clearly evident that, apply our concepts as we will, without the help of intuition, by mere conceptual division into elements, we can never discover a sum."[26]

Couturat retorts upon Kant that it is practically impossible to have precise and complete intuitions of numbers of the order of millions, and that these could never be calculated exactly if recourse to intuition were necessary. "What is true of the large numbers," he continues, "is true also of the small, and consequently it is not intuition but reason that enables us to say that 2 and 2 make 4."[27] Evidently Couturat overlooks the case of the phenomenal calculator who handles millions as the average mortal handles units, and without being able to analyze the process; or again, the undoubted fact that the average civilized man would be a mathematical prodigy to the average primitive. And again, it is not easy to see that there is a more excessive dogmatism in assuming that our intuitions of the great numbers are in character with our intuitions of the small, than in asserting that because we have no intuitions of the great (supposing this true) we can therefore have none of the small,—which is Couturat's position.

Nevertheless, there is justice in Couturat's criticisms,

[25] *K. d. r. V.*, 205-6.
[26] *Ibid.*, 15-16.
[27] L. Couturat, *Les principes des mathématiques*, p. 256.

especially to the effect that Kant's notion of "numerical formulas," calling as it does for an infinity of irreducible synthetic insights, ill conforms to our notion of rationality, and is, indeed, only a masked intrusion of the old empirical view of number. The difficulty with Kant's view is that the number syntheses reduce to no law, which offends our sense of the reasonable, hyper-conscious as it is when touched on the side of mathematics. Kant's *a priori* synthesis is after all only a designation, and, as Poincaré says, to christen a difficulty is not to solve it.

Poincaré's own view—which may be described as Kantian with a saving salt of empiricism—is an interesting variation. The foundation of the idea of number is mathematical induction, and the essence of mathematical induction is reasoning by recurrence, while reasoning by recurrence has for its proper character just that "it contains, as it were condensed into a single formula, an infinity of syllogisms." Such a rule cannot come to us from experience; experience can show it to hold for a limited portion, but only for a limited portion, of the endless suite of numbers. If it were a matter only of this limited portion the principle of contradiction would suffice, permitting us to develop as many syllogisms as we wish; but when it comes to embracing an infinity in a single formula, when the infinite is in question, then this principle fails, and it is just here too that experience is impotent. The rule of recurrence, "inaccessible alike to analytic demonstration and to experience, is the veritable type of the synthetic judgment *a priori*." [28]

Why, then, Poincaré asks, does such a form of judgment impose itself upon us so irresistibly? "Because it

[28] *La science et l'hypothèse,* Chap. I. Cf. p. 37: "Nous avons la faculté de concevoir qu'une unité peut être ajoutée à une collection d'unités; c'est grâce à l'expérience que nous avons l'occasion d'exercer cette faculté et que nous en prenons conscience: mais, dès ce moment, nous sentons que notre pouvoir n'a pas de limite et que nous pourrions compter indéfiniment, quoique nous n'ayons jamais eu à compter qu'un numbre d'objets."

is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of an act once this act is found possible. The mind has a direct intuition of this power; experience can be only an occasion for making use of it and hence of becoming conscious of it."

But there is another and an important feature of reasoning by recurrence which Poincaré emphasizes, and this is the *inventive* character of its judgments. They are not only intuitive, born of the nature of the mind, they are also creative; and indeed it is mathematical induction alone which can apprise us of the new. Each number, then, is to be looked upon as an invention—not due to physical experience, but a self-discovery of the mind. But invention and the self-discovery of the mind do not cease so long as life lasts; and so, says Poincaré in another connection,[29] "when I speak of all the whole numbers, I mean by that all the whole numbers that have been discovered and will one day be discovered.... And it is just this *possibility* of discovery that is the infinite."

The psychological temper of this view is apparent; in so far it is empirical. But the validity of mathematical judgments is independent of the vagaries of experience; it is derived from the structure of the mind rather than from the accidents of a conscious life, and in so far the judgments are *a priori* and metempirical. Whether mathematical truths represent not only the organization of mind but also the organization of nature is an epistemological question for which Poincaré suggests an interesting answer, but it is properly a question, not of mathematics, but of metaphysics.

Analogous to Poincaré's view is that of Bergson, which also must be regarded as Kantian in type. Bergson begins his analysis of the number concept with the categories of

[29] *Dernières pensées*, p. 131.

unity and multiplicity: every individual number is to be regarded as a ratio between the one and the many, unit and totality. "There are two species of unity," writes Bergson,[30] "the one definitive, which will form a number in adding itself to itself; the other provisional, that of this number which, in itself multiple, borrows its unity from the simple act by which the intelligence perceives it. And it is undeniable that when we image to ourselves the unitary components of the number we believe ourselves to be thinking of indivisibles, this belief entering as a considerable factor in the notion that we can conceive number apart from space. In every case, viewing the matter more nearly, we shall see that each unity is that of a simple act of the mind, and that, this act consisting in uniting, it is necessary that some multiplicity serve as its matter."

The two poles of the idea of number, unity and multiplicity, correspond in Bergson's view to the subjective and objective elements of experience,—ultimately and respectively to time and space, use and generation. "You can never draw from an idea which you have constructed more than you have put into it, and if the unity with which you compose your number is the unity of an act and not of an object, no effort of analysis can evoke from it more than unity pure and simple. Without doubt when you equate the number 3 to the sum of $1 + 1 + 1$, nothing prevents you from holding as indivisible the units which compose it, but this is because you do not utilize the multiplicity with which each of these units is big. It is, moreover, probable that the number 3 presents itself to our mind in this simple form, because we are thinking rather of the manner in which we obtained it than of the use we can make of it. But we ought to see that if all multiplication implies the possibility of treating any number soever as a provisional unity which will add itself to itself, in-

[30] *Les données immédiates de la conscience*, pp. 58-65.

versely the units in their turn are veritable numbers as great as one may wish, though one provisionally assumes them to be indecomposable in order to combine them *inter se*. Moreover, by the very fact that the possibility of dividing unity into as many parts as are desired is admitted it is regarded as extended." In fine: "What properly pertains to the mind is the indivisible process by which it fixes its attention successively upon the diverse parts of a given space; but the parts thus isolated are conserved in order to be added to others, and once added among themselves they are open to a new decomposition of whatever sort. They are then parts of space, and space is the matter with which the mind constructs number, the milieu in which the mind places it."

Thus in the Bergsonian view numbers are ratios mediating time and space. The order in which they fall is first of all the order in which achieved experience presents itself, i .e., it is spatial. But space-perceptions are all provisional in character; consequently numbers are all provisional in character. Numerical order is not continuous, but composed *per saltum* (*par sauts brusques*); we form our numbers turn by turn, each assuming the character of a mathematical point separated by an interval of space from the point following, but as we recede in our series from the points first formed these tend to unite into a line, their synthesis being the necessary consequence of our averted attention. But "once formed according to a determinate law, the number is decomposable according to any law whatever"; and here we reach the apparent freedom and apriority of the mathematical reason, a number in course of formation is not the same as a number once formed; it is only the latter that is really divisible.

Doubtless to minds enamored of the eternal, Bergson's view will seem a veritable anarchy; perhaps metaphysically it is so; but it can hardly be denied that it gives a

fair description of the manner in which we actually learn and apply our numbers, and it gives also an intelligibility to the old-fashioned notion that number is generated by successive acts of attention which the old-fashioned explanations do not possess. This is due, of course, to the assumption of an intuitive reason, differing from Kant's—as with Poincaré—chiefly in its more direct reliance upon the course of conscious events, upon psychology conceived as mental history.

Nor is it altogether fanciful to see in Bergson's view a striking analogue of Plato's. Like Plato he conceives number as essentially a ratio. Like Plato he conceives the realm of numbers as a median realm, uniting the one and the many, participating in the one direction in the essential unity of thought, in the other expressing itself as the multiplicity of things. Number is the category which unites subjective and objective, ideal and material,—or in Bergsonian terms, time and space.

V.

The types of definition of number which we have been considering raise certain inevitable issues—none more inevitable than the question of the relation of psychology to logic, and of both these sciences to epistemology.

If we contrast the older empirical conception of number with the logistic view, we see at once that the former defines number from the point of departure of number genesis while the latter analyzes its nature irrespective of its origins. From this we may guess both the reason for the dependence of the older conception upon the act of counting, in the definition of number, and the reason for the aversion to counting (for their denials of its significance amounts to this) on the part of the logisticians. For there can be no question that, historically considered, the invention of counting is the beginning of a science of number;

nor again, that a study of the number-systems of primitive peoples, and indeed of the civilized, yield a direct insight into the modes in which numbers are thought. The psychology of number-consciousness is, therefore, a direct key to our mathematical use of numbers.

But is there another and more efficient key, not perhaps explaining the nature of our consciousness of numbers, but explaining why they are found to be applicable to experience or even susceptible of metempirical developments? To this question the logisticians respond with a various affirmative, "various" because, while for some logistic is a purely nominalistic science (or, more correctly, purely symbolistic), for others it is the clue to a realism transcending the fictions which impair all empirically-originated speech.

It must be owned that there is a kind of experiential warrant for each of these views—the Uranian as well as the Pandemian. For if the latter can appeal to the universal conformity of number notions in process of formation, to our physical and mental structure and needs, the Uranian reason can retort with the universal and seemingly superhuman validity of mathematics. Mathematical demonstrations need only to be understood in order to be convincing, and if there be such a thing as infallibility there can be no test for it save this. From such infallibility the Uranian may infer, with a show of force, that number is not the product of our experience, but is imposed on us by the structure of the universe. Mathematical truth is, at all events, more universal than anything else we know.

But this is a doctrine, not of logic, but of epistemology. Uranians do not like the word,—it has psychological associations. They prefer to mark their own science as at once hyper-psychological, hyper-epistemological, hyperlogical—a science which can have no name, since every name is contaminated with the experiential (humanly ex-

periential) references of language. They aim rather at a system of symbols which shall be untalkable, though catholic of the meanings of speech as well as of all other meanings;[31] they would introduce us into a sphere where human relations and merely human thinking are merged into the crystalline structure of the de-reified reality of a cosmos transcending speech.

"Extravagant realism" is the only historic caption that can fit this point of view, and extravagant realism is the philosophical creed which Russell at least is ready to make his own.[32] That some adherents of the movement balk at this is no matter of surprise; but surely it is with ill reason, for the philosophic alternative which is left them is a nominalism without even the consolations of speech. When symbols are refined to such an extent that they are but the symbols of systems of unutterable ideas, whose generality outgeneralizes nature, then surely their inventors are worse than dumb; they have become cousin-german to the apostles of the flux, and, with Cratylus, nothing is left them but to wag impotent digits.

When the rigorous following out of the mathematical reason leads to such extreme views, we may well bear in mind M. Milhaud's caution against a too naive confidence in the dogmatisms of our understanding. We may well ask by what right (since it is from no definable experience) transcendental realism justifies its *ex cathedra* affirmations; or, with Poincaré, what value is to be attached to a symbolism so ineffable that no testimony of familiar fact can sustain it. And we will surely be led to inquire if there be not some secure middle way, satisfying at once to our reason and our sense.

[31] I can imagine no more downright statement of the point of view than that of A. Padoa (*Bibliothèque du Congrès International de Philosophie*, 1901, Vol. III, pp. 317f.). Surely, when we are told that science is the peril of logic, that reasoning in order to be safe must be empty, we may well draw heretical breaths!

[32] *Monist*, October, 1914.

Now it would be presumption to affirm that the Kantian view—which we might term the "moderate realism" of the development—even as amended by Poincaré and Bergson is wholly satisfying. There are unquietable difficulties besetting every relativism, and these become accentuated when the relativity is between such extreme factors as reason and sensibility. It is far more comfortable to fashion a shapely abode of ideas of a single order and name it intellect than to be faithful to all the factors that enter into the cognizable world; nevertheless, it is only with this inclusive faithfulness at once to fact and to reason that temperaments of a certain kind can find their rest.

Herein is the merit of the neo-Kantian view. It sees the crudities of the old naive empiricism quite as clearly as do the logisticians; but for all that it is unwilling to abandon empirical leadings or to deny the centrality of our human experience, for mathematical as for all other meanings. Indeed, it asks, and asks fairly, of the logisticians by what right they assume that the numbers and measures that tell and mete the physical wôrld are only illustrative cases to be subsumed under some cosmic Number, super-human and supra-mundane. Why, for example, is "the suite of *natural* numbers" so named, and why made the model for the conceptualization of all other series, if it be not due to some greater intimacy of nature which number has with this suite than with the others?

Referring to the arithmetical definition of continuity Poincaré says:[33] "This definition makes a ready disposal of the intuitive origin of the notion of continuity, and of all the riches which this notion conceals. It returns to the type of those definitions—so frequent in mathematics since the tendency to arithmetize this science—definitions mathematically sound, but philosophically unsatisfying. They replace the objects to be defined and the intuitive notion

[33] *Dernières pensées*, p. 65.

of this object by a construction made of simpler materials; one sees indeed that one can effectively make this construction with these materials, but one sees also that one can make many others. What is not to be seen is the deeper reason why one assembles these materials in just this, and not in another fashion." And again:[34] "Among all the constructions that one can make with the materials furnished by logic, a choice must be made; the true geometer makes this choice judiciously because he is guided by a sure instinct, or by some vague consciousness of I know not what geometry more profound and more hidden, which alone makes the value of the edifice built."

There is a sense, as we have said, in which the world *is* all possible worlds; but there is a commoner and more valuable sense according to which the world we call real is only one among many possible worlds. The problem at once of philosophy and of all rational life is to tell us just what this unique reality is, why the materials of creation have been assembled in just this, and not in another fashion.

Both Poincaré and Bergson recognize in mathematical reasoning a power or enterprise of the spirit which is in some sense prior to experience. It is in this that they are Kantians. This power, or intuition as they agree in calling it, gives to mathematical truths their sanctioning validity. But the validity of mathematics is not supposed, as with the logisticians, to derive from a firmament above the firmament; it holds only within the ranges of human insight, and indeed it is the definition of the utmost reach of this insight. "When I speak of all the whole numbers, I mean by that all the whole numbers that have been discovered and will one day be discovered....And it is just this possibility of discovery that is the infinite," says Poincaré. If I read Bergson aright, I judge his conception of

[34] *Science et méthode*, p. 158.

the unity of living time, within which number is generated in the perception of differences, to be not radically divergent from Poincaré's meaning; and certainly their common view squares with the kind of interpretation which language can give of number, and which the ordinarily thoughtful intelligence can accept.

Nor do I hesitate to add that its metaphysical implications are rich and profound. For a view of number which, while holding it within the leash of human experience makes of it the measure of our expectation of life, is surely sufficiently grandiose for any imagination, if it seem to make that expectation infinite. The intuition which gives the sanction becomes the testimony to a truth in number transcending the facts to which it is applied—that is, the little range of life here present—though not transcending the possibilities of real experience. Plato found in mathematical intuitions recollections from a previous life of the intelligence, Bergson and Poincaré treat them rather as prophecies of life to come; but these are only variations of a common doctrine.

Hartley Burr Alexander.

University of Nebraska.

THE ULTIMATE CONSTITUENTS OF MATTER.*

I WISH to discuss in this article no less a question than the ancient metaphysical query, "What is matter?" The question, "What is matter?" in so far as it concerns philosophy, is, I think, already capable of an answer which in principle will be as complete as an answer can hope to be; that is to say, we can separate the problem into an essentially soluble and an essentially insoluble portion, and we can now see how to solve the essentially soluble portion, at least as regards its main outlines. It is these outlines which I wish to suggest in the present article. My main position, which is realistic, is, I hope and believe, not remote from that of Professor Alexander, by whose writings on this subject I have profited greatly.[1] It is also in close accord with that of Dr. Nunn.[2]

Common sense is accustomed to the division of the world into mind and matter. It is supposed by all who have never studied philosophy that the distinction between mind and matter is perfectly clear and easy, that the two do not at any point overlap, and that only a fool or a philosopher could be in doubt as to whether any given entity is mental or material. This simple faith survives in Descartes and in a somewhat modified form in Spinoza, but with Leibniz it begins to disappear, and from his day to our own almost every philosopher of note has criticized and rejected the dualism of common sense. It is my intention in this article to defend this dualism; but before defending

* An address delivered to the Philosophical Society of Manchester in February 1915.

[1] Cf. especially Samuel Alexander, "The Basis of Realism," *British Academy*, Vol. VI.

[2] "Are Secondary Qualities Independent of Perception?" *Proc. Arist. Soc.*, 1909-10, pp. 151-218.

it we must spend a few moments on the reasons which have prompted its rejection.

Our knowledge of the material world is obtained by means of the senses, of sight and touch and so on. At first it is supposed that things are just as they seem, but two opposite sophistications soon destroy this naive belief. On the one hand the physicists cut up matter into molecules, atoms, corpuscles, and as many more such subdivisions as their future needs may make them postulate, and the units at which they arrive are uncommonly different from the visible, tangible objects of daily life. A unit of matter tends more and more to be something like an electromagnetic field filling all space, though having its greatest intensity in a small region. Matter consisting of such elements is as remote from daily life as any metaphysical theory. It differs from the theories of metaphysicians only in the fact that its practical efficacy proves that it contains some measure of truth and induces business men to invest money on the strength of it; but in spite of its connection with the money market, it remains a metaphysical theory none the less.

The second kind of sophistication to which the world of common sense has been subjected is derived from the psychologists and physiologists. The physiologists point out that what we see depends upon the eye, that what we hear depends upon the ear, and that all our senses are liable to be affected by anything which affects the brain, like alcohol or hasheesh. Psychologists point out how much of what we think we see is supplied by association or unconscious inference, how much is mental interpretation, and how doubtful is the residuum which can be regarded as crude datum. From these facts it is argued by the psychologists that the notion of a datum passively received by the mind is a delusion, and it is argued by the physiologists that even if a pure datum of sense could be obtained by

the analysis of experience, still this datum could not belong, as common sense supposes, to the outer world, since its whole nature is conditioned by our nerves and sense organs, changing as they change in ways which it is thought impossible to connect with any change in the matter supposed to be perceived. This physiologist's argument is exposed to the rejoinder, more specious than solid, that our knowledge of the existence of the sense organs and nerves is obtained by that very process which the physiologist has been engaged in discrediting, since the existence of the nerves and sense organs is only known through the evidence of the senses themselves. This argument may prove that some reinterpretation of the results of physiology is necessary before they can acquire metaphysical validity. But it does not upset the physiological argument in so far as this constitutes merely a *reductio ad absurdum* of naive realism.

These various lines of argument prove, I think, that some part of the beliefs of common sense must be abandoned. They prove that, if we take these beliefs as a whole, we are forced into conclusions which are in part self-contradictory; but such arguments cannot of themselves decide what portion of our common-sense beliefs is in need of correction. Common sense believes that what we see is physical, outside the mind, and continuing to exist if we shut our eyes or turn them in another direction. I believe that common sense is right in regarding what we see as physical and outside the mind, but is probably wrong in supposing that it continues to exist when we are no longer looking at it. It seems to me that the whole discussion of matter has been obscured by two errors which support each other. The first of these is the error that what we see, or perceive through any of our other senses, is subjective: the second is the belief that what is physical must be persistent. Whatever physics may regard as the ultimate

constituents of matter, it always supposes these constituents to be indestructible. Since the immediate data of sense are not indestructible but in a state of perpetual flux, it is argued that these data themselves cannot be among the ultimate constituents of matter. I believe this to be a sheer mistake. The persistent particles of mathematical physics I regard as logical constructions, symbolic fictions enabling us to express compendiously very complicated assemblages of facts; and, on the other hand, I believe that the actual data in sensation, the immediate objects of sight or touch or hearing, are extra-mental, purely physical, and among the ultimate constituents of matter.

My meaning in regard to the impermanence of physical entities may perhaps be made clearer by the use of Bergson's favorite illustration of the cinematograph. When I first read Bergson's statement that the mathematician conceives the world after the analogy of a cinematograph, I had never seen a cinematograph, and my first visit to one was determined by the desire to verify Bergson's statement, which I found to be completely true, at least so far as I am concerned. When, in a picture palace, we see a man rolling down hill, or running away from the police, or falling into a river, or doing any of those other things to which men in such places are addicted, we know that there is not really only one man moving, but a succession of films, each with a different momentary man. The illusion of persistence arises only through the approach to continuity in the series of momentary men. Now what I wish to suggest is that in this respect the cinema is a better metaphysician than common sense, physics, or philosophy. The real man too, I believe, however the police may swear to his identity, is really a series of momentary men, each different one from the other, and bound together, not by a numerical identity, but by continuity and certain intrinsic causal laws. And what applies to men applies

equally to tables and chairs, the sun, moon and stars. Each of these is to be regarded, not as one single persistent entity, but as a series of entities succeeding each other in time, each lasting for a very brief period, though probably not for a mere mathematical instant. In saying this I am only urging the same kind of division in time as we are accustomed to acknowledge in the case of space. A body which fills a cubic foot will be admitted to consist of many smaller bodies, each occupying only a very tiny volume; similarly a thing which persists for an hour is to be regarded as composed of many things of less duration. A true theory of matter requires a division of things into time-corpuscles as well as into space-corpuscles.

The world may be conceived as consisting of a multitude of entities arranged in a certain pattern. The entities which are arranged I shall call "particulars." The arrangement or pattern results from relations among particulars. Classes or series of particulars, collected together on account of some property which makes it convenient to be able to speak of them as wholes, are what I call logical constructions or symbolic fictions. The particulars are to be conceived, not on the analogy of bricks in a building, but rather on the analogy of notes in a symphony. The ultimate constituents of a symphony (apart from relations) are the notes, each of which lasts only for a very short time. We may collect together all the notes played by one instrument: these may be regarded as the analogues of the successive particulars which common sense would regard as successive states of one "thing." But the "thing" ought to be regarded as no more "real" or "substantial" than, for example, the rôle of the trombone. As soon as "things" are conceived in this manner it will be found that the difficulties in the way of regarding immediate objects of sense as physical have largely disappeared.

When people ask, "Is the object of sense mental or physical?" they seldom have any clear idea either what is meant by "mental" or "physical," or what criteria are to be applied for deciding whether a given entity belongs to one class or the other. I do not know how to give a sharp definition of the word "mental," but something may be done by enumerating occurrences which are indubitably mental: believing, doubting, wishing, willing, being pleased or pained, are certainly mental occurrences; so are what we may call experiences, seeing, hearing, smelling, perceiving generally. But it does not follow from this that what is seen, what is heard, what is smelt, what is perceived, must be mental. When I see a flash of lightning, my seeing of it is mental, but what I see, although it is not quite the same as what anybody else sees at the same moment, and although it seems very unlike what the physicist would describe as a flash of lightning, is not mental. I maintain, in fact, that if the physicist could describe truly and fully all that occurs in the physical world when there is a flash of lightning, it would contain as a constituent what I see, and also what is seen by anybody else who would commonly be said to see the same flash. What I mean may perhaps be made plainer by saying that if my body could remain in exactly the same state in which it is, although my mind had ceased to exist, precisely that object which I now see when I see the flash would exist, although of course I should not see it, since my seeing is mental. The principal reasons which have led people to reject this view have, I think, been two: first, that they did not adequately distinguish between my seeing and what I see; secondly, that the causal dependence of what I see upon my body has made people suppose that what I see cannot be "outside" me. The first of these reasons need not detain us, since the confusion only needs to be pointed out in order to be obviated; but the second requires some discussion, since it

can only be answered by removing current misconceptions, on the one hand as to the nature of space, and on the other, as to the meaning of causal dependence.

When people ask whether colors, for example, or other secondary qualities are inside or outside the mind, they seem to suppose that their meaning must be clear, and that it ought to be possible to say yes or no without any further discussion of the terms involved. In fact, however, such terms as "inside" or "outside" are very ambiguous. What is meant by asking whether this or that is "in" the mind? The mind is not like a bag or a pie; it does not occupy a certain region in space, or, if it does, what is in that region is presumably part of the brain, which would not be said to be in the mind. When people say that sensible qualities are in the mind, they do not mean "spatially contained in" in the sense in which the blackbirds were in the pie. We might regard the mind as an assemblage of particulars, namely what would be called "states of mind," which would belong together in virtue of some specific common quality. The common quality of all states of mind would be the quality designated by the word "mental"; and besides this we should have to suppose that each separate person's states of mind have some common characteristic distinguishing them from the states of mind of other people. Ignoring this latter point, let us ask ourselves whether the quality designated by the word "mental" does, as a matter of observation, actually belong to objects of sense, such as colors or noises. I think any candid person must reply that, however difficult it may be to know what we mean by "mental," it is not difficult to see that colors and noises are not mental in the sense of having that intrinsic peculiarity which belongs to beliefs and wishes and volitions, but not to the physical world. Berkeley advances on this subject a plausible argument[3] which seems to me to rest upon

[3] First dialogue between Hylas and Philonous, *Works* (Fraser's edition), I, p. 266; *Three Dialogues* (published by Open Court Pub. Co.), p. 15.

an ambiguity in the word "pain." He argues that the realist supposes the heat which he feels in approaching a fire to be something outside his mind, but that as he approaches nearer and nearer to the fire the sensation of heat passes imperceptibly into pain, and that no one could regard pain as something outside the mind. In reply to this argument, it should be observed in the first place that the heat of which we are immediately aware is not in the fire but in our own body. It is only by inference that the fire is judged to be the cause of the heat which we feel in our body. In the second place (and this is the more important point), when we speak of pain we may mean one of two things: we may mean the object of the sensation or other experience which has the quality of being painful, or we may mean the quality of painfulness itself. When a man says he has a pain in his great toe, what he means is that he has a sensation associated with his great toe and having the quality of painfulness. The sensation itself, like every sensation, consists in experiencing a sensible object, and the experiencing has that quality of painfulness which only mental occurrences can have, but which may belong to thoughts or desires, as well as to sensations. But in common language we speak of the sensible object experienced in a painful sensation as a pain, and it is this way of speaking which causes the confusion upon which the plausibility of Berkeley's argument depends. It would be absurd to attribute the quality of painfulness to anything non-mental, and hence it comes to be thought that what we call a pain in the toe must be mental. In fact, however, it is not the sensible object in such a case which is painful, but the sensation, that is to say, the experience of the sensible object. As the heat which we experience from the fire grows greater, the experience passes gradually from being pleasant to being painful, but neither the pleasure nor the pain is a quality of the object experienced

as opposed to the experience, and it is therefore a fallacy to argue that this object must be mental on the ground that painfulness can only be attributed to what is mental.

If, then, when we say that something is in the mind we mean that it has a certain recognizable intrinsic characteristic such as belongs to thoughts and desires, it must be maintained on grounds of immediate inspection that objects of sense are not in any mind.

A different meaning of "in the mind" is however to be inferred from the arguments advanced by those who regard sensible objects as being in the mind. The arguments used are, in the main, such as would prove the causal dependence of objects of sense upon the percipient. Now the notion of causal dependence is very obscure and difficult, much more so in fact than is generally realized by philosophers. I shall return to this point in a moment. For the present however, accepting the notion of causal dependence without criticism, I wish to urge that the dependence in question is rather upon our bodies than upon our minds. The visual appearance of an object is altered if we shut one eye, or squint, or look previously at something dazzling; but all these are bodily acts, and the alterations which they effect are to be explained by physiology and optics, not by psychology.[4] They are in fact of exactly the same kind as the alterations effected by spectacles or a microscope. They belong therefore to the theory of the physical world, and can have no bearing upon the question whether what we see is causally dependent upon the mind. What they do tend to prove, and what I for my part have no wish to deny, is that what we see is causally dependent upon our body and is not, as crude common sense would suppose, something which would exist equally if our eyes and nerves and brain were absent, any more than the visual appearance presented by an object seen through a microscope would re-

[4] This point has been well urged by the American realists.

main if the microscope were removed. So long as it is supposed that the physical world is composed of stable and more or less permanent constituents, the fact that what we see is changed by changes in our body appears to afford reason for regarding what we see as not an ultimate constituent of matter. But if it is recognized that the ultimate constituents of matter are as circumscribed in duration as in spatial extent, the whole of this difficulty vanishes.

There remains however another difficulty, connected with space. When we look at the sun we wish to know something about the sun itself, which is ninety-three million miles away; but what we see is dependent upon our eyes, and it is difficult to suppose that our eyes can affect what happens at a distance of ninety-three million miles. Physics tells us that certain electromagnetic waves start from the sun, and reach our eyes after about eight minutes. They there produce disturbances in the rods and cones, thence in the optic nerve, thence in the brain. At the end of this purely physical series, by some odd miracle, comes the experience which we call "seeing the sun," and it is such experiences which form the whole and sole reason for our belief in the optic nerve, the rods and cones, the ninety-three million miles, the electromagnetic waves, and the sun itself. It is this curious oppositeness of direction between the order of causation as affirmed by physics, and the order of evidence as revealed by theory of knowledge, that causes the most serious perplexities in regard to the nature of physical reality. Anything that invalidates our seeing, as a source of knowledge concerning physical reality, invalidates also the whole of physics and physiology. And yet, starting from a common-sense acceptance of our seeing, physics has been led step by step to the construction of the causal chain in which our seeing is the last link, and the immediate object which we see cannot be regarded as

that initial cause which we believe to be ninety-three million miles away, and which we are inclined to regard as the "real" sun.

I have stated this difficulty as forcibly as I can, because I believe that it can only be answered by a radical analysis and reconstruction of all the conceptions upon whose employment it depends.

Space, time, matter and cause, are the chief of these conceptions. Let us begin with the conception of cause.

Causal dependence, as I observed a moment ago, is a conception which it is very dangerous to accept at its face value. There exists a notion that in regard to any event there is something which may be called *the* cause of that event—some one definite occurrence, without which the event would have been impossible and with which it becomes necessary. An event is supposed to be dependent upon its cause in some way in which it is not dependent upon other things. Thus men will urge that the mind is dependent upon the brain, or, with equal plausibility, that the brain is dependent upon the mind. It seems not improbable that if we had sufficient knowledge we could infer the state of a man's mind from the state of his brain, or the state of his brain from the state of his mind. So long as the usual conception of causal dependence is retained, this state of affairs can be used by the materialist to urge that the state of our brain causes our thoughts, and by the idealist to urge that our thoughts cause the state of our brain. Either contention is equally valid or equally invalid. The fact seems to be that there are many correlations of the sort which may be called causal, and that, for example, either a physical or a mental event can be predicted, theoretically, either from a sufficient number of physical antecedents or from a sufficient number of mental antecedents. To speak of *the* cause of an event is therefore misleading. Any set of antecedents from which

the event can theoretically be inferred by means of correlations might be called *a* cause of the event. But to speak of *the* cause is to imply a uniqueness which does not exist.

The relevance of this to the experience which we call "seeing the sun" is obvious. The fact that there exists a chain of antecedents which makes our seeing dependent upon the eyes and nerves and brain does not even tend to show that there is not another chain of antecedents in which the eyes and nerves and brain as physical things are ignored. If we are to escape from the dilemma which seemed to arise out of the physiological causation of what we see when we say we see the sun, we must find, at least in theory, a way of stating causal laws for the physical world, in which the units are not material things, such as the eyes and nerves and brain, but momentary particulars of the same sort as our momentary visual object when we look at the sun. The sun itself and the eyes and nerves and brain must be regarded as assemblages of momentary particulars. Instead of supposing, as we naturally do when we start from an uncritical acceptance of the apparent dicta of physics, that *matter* is what is "really real" in the physical world, and that the immediate objects of sense are mere phantasms, we must, instead, regard matter as a logical construction, of which the constituents will be just such evanescent particulars as may, when an observer happens to be present, become data of sense to that observer. What physics regards as the sun of eight minutes ago will be a whole assemblage of particulars, existing at different times, spreading out from a center with the velocity of light, and containing among their number all those visual data which are seen by people who are now looking at the sun. Thus the sun of eight minutes ago is a class of particulars, and what I see when I now look at the sun is one member of this class. The various particulars constituting this class will be correlated with each other by a

certain continuity and certain intrinsic laws of variation as we pass outwards from the center, together with certain modifications correlated extrinsically with other particulars which are not members of this class. It is these extrinsic modifications which represent the sort of facts that, in our former account, appeared as the influence of the eyes and nerves in modifying the appearance of the sun.[5]

The *prima facie* difficulties in the way of this view are chiefly derived from an unduly conventional theory of space. It might seem at first sight as if we had packed the world much fuller than it could possibly hold. At every place between us and the sun, we said, there is to be a particular which is to be a member of the sun as it was a few minutes ago. There will also, of course, have to be a particular which is a member of any planet or fixed star that may happen to be visible from that place. At the place where I am, there will be particulars which will be members severally of all the "things" I am now said to be perceiving. Thus throughout the world, everywhere, there will be an enormous number of particulars coexisting in the same place. But these troubles result from contenting ourselves too readily with the merely three-dimensional space to which schoolmasters have accustomed us. The space of the real world is a space of six dimensions, and as soon as we realize this we see that there is plenty of room for all the particulars for which we want to find positions. In order to realize this we have only to return for a moment from the polished space of physics to the rough and untidy space of our immediate sensible experience. The space of one man's sensible objects is a three-dimensional space. It does not appear probable that two men ever both perceive at the same time any one sensible object; when they are said to see the same thing

[5] Cf. T. P. Nunn, "Are Secondary Qualities Independent of Perception?" *Proc. Aris. Soc.*, 1909-1910.

or hear the same noise, there will always be some difference, however slight, between the actual shapes seen or the actual sounds heard. If this is so, and if, as is generally assumed, position in space is purely relative, it follows that the space of one man's objects and the space of another man's objects have no place in common, that they are in fact different spaces, and not merely different parts of one space. I mean by this that such immediate spatial relations as are perceived to hold between the different parts of the sensible space perceived by one man, do not hold between parts of sensible spaces perceived by different men. There are therefore a multitude of three-dimensional spaces in the world: there are all those perceived by observers, and presumably also those which are not perceived, merely because no observer is suitable situated for perceiving them.

But although these spaces do not have to one another the same kind of spatial relations as obtain between the parts of one of them, it is nevertheless possible to arrange these spaces themselves in a three-dimensional order. This is done by means of the correlated particulars which we regard as members (or aspects) of one physical thing. When a number of people are said to see the same object, those who would be said to be near to the object see a particular occupying a larger part of their field of vision than is occupied by the corresponding particular seen by people who would be said to be farther from the thing. By means of such considerations it is possible, in ways which need not now be further specified, to arrange all the different spaces in a three-dimensional series. Since each of the spaces is itself three-dimensional, the whole world of particulars is thus arranged in a six-dimensional space, that is to say, six coordinates will be required to assign completely the position of any given particular, namely three to assign its position in its own space and three more to assign the position of its space among the other spaces.

There are two ways of classifying particulars: we may take together all those that belong to a given "perspective," or all those that are, as common sense would say, different "aspects" of the same "thing." For example, if I am (as is said) seeing the sun, what I see belongs to two assemblages: (1) the assemblage of all my present objects of sense, which is what I call a "perspective"; (2) the assemblage of all the different present particulars which would be called aspects of the sun—this assemblage is what I define as *being* the sun at the present time. Thus "perspectives" and "things" are merely two different ways of classifying particulars. It is to be observed that there is no *a priori* necessity for particulars to be susceptible of this double classification. There may be what might be called "wild" particulars, not having the usual relations by which the classification is effected; perhaps dreams and hallucinations are composed of particulars which are "wild" in this sense.

The exact definition of what is meant by a perspective is not quite easy. So long as we confine ourselves to visible objects or to objects of touch we might define the perspective of a given particular as "all particulars which have a simple (direct) spatial relation to the given particular." Between two patches of color which I see now, there is a direct spatial relation which I equally see. But between patches of color seen by different men there is only an indirect constructed spatial relation by means of the placing of "things" in physical space (which is the same as the space composed of perspectives). Those particulars which have direct spatial relations to a given particular will belong to the same perspective. But if, for example, the sounds which I hear are to belong to the same perspective with the patches of color which I see, there must be particulars which have no direct spatial relation and yet belong to the same perspective. We cannot define a perspective as all the data of one percipient at one time,

because we wish to allow the possibility of perspectives which are not perceived by any one. There will be need, therefore, in defining a perspective, of some principle derived neither from psychology nor from space.

Such a principle may be obtained from the consideration of *time*. The one all-embracing time, like the one all-embracing space, is a construction; there is no *direct* time-relation between particulars belonging to my perspective and particulars belonging to another man's. On the other hand, any two particulars of which I am aware are either simultaneous or successive, and their simultaneity or successiveness is sometimes itself a datum to me. We may therefore define the perspective to which a given particular belongs as "all particulars simultaneous with the given particular," where "simultaneous" is to be understood as a direct simple relation, not the derivative constructed relation of physics. It may be observed that the introduction of "local time" suggested by the principle of relativity has effected, for purely scientific reasons, much the same multiplication of times as we have just been advocating.

The sum-total of all the particulars that are (directly) either simultaneous with or before or after a given particular may be defined as the "biography" to which that particular belongs. It will be observed that, just as a perspective need not be actually perceived by any one, so a biography need not be actually lived by any one. Those biographies that are lived by no one are called "official."

The definition of a "thing" is effected by means of continuity and of correlations which have a certain differential independence of other "things." That is to say, given a particular in one perspective, there will usually in a neighboring perspective be a very similar particular, differing from the given particular, to the first order of small quantities, according to a law involving only the difference

of position of the two perspectives in perspective space, and not any of the other "things" in the universe. It is this continuity and differential independence in the law of change as we pass from one perspective to another that defines the class of particulars which is to be called "one thing."

Broadly speaking, we may say that the physicist finds it convenient to classify particulars into "things," while the psychologist finds it convenient to classify them into "perspectives" and "biographies," since one perspective *may* constitute the momentary data of one percipient, and one biography *may* constitute the whole of the data of one percipient throughout his life.

We may now sum up our discussion. Our object has been to discover as far as possible the nature of the ultimate constituents of the physical world. When I speak of the "physical world" I mean, to begin with, the world dealt with by physics. It is obvious that physics is an empirical science, giving us a certain amount of knowledge and based upon evidence obtained through the senses. But partly through the development of physics itself, partly through arguments derived from physiology, psychology or metaphysics, it has come to be thought that the immediate data of sense could not themselves form part of the ultimate constituents of the physical world, but were in some sense "mental," "in the mind," or "subjective." The grounds for this view, in so far as they depend upon physics, can only be adequately dealt with by rather elaborate constructions depending upon symbolic logic, showing that out of such materials as are provided by the senses it is possible to construct classes and series having the properties which physics assigns to matter. Since this argument is difficult and technical, I have not embarked upon it in this article. But in so far as the view that sense-data are "mental" rests upon physiology, psychology, or metaphysics, I have

tried to show that it rests upon confusions and prejudices —prejudices in favor of permanence in the ultimate constituents of matter, and confusions derived from unduly simple notions as to space, from the causal correlation of sense-data with sense-organs, and from failure to distinguish between sense-data and sensations. If what we have said on these subjects is valid, the existence of sense-data is logically independent of the existence of mind, and is causally dependent upon the *body* of the percipient rather than upon his mind. The causal dependence upon the body of the percipient, we found, is a more complicated matter than it appears to be, and, like all causal dependence, is apt to give rise to erroneous beliefs through misconceptions as to the nature of causal correlation. If we have been right in our contentions, sense-data are merely those among the ultimate constituents of the physical world, of which we happen to be immediately aware; they themselves are purely physical, and all that is mental in connection with them is our awareness of them, which is irrelevant to their nature and to their place in physics.

Unduly simple notions as to space have been a great stumbling-block to realists. When two men look at the same table, it is supposed that what the one sees and what the other sees are in the same place. Since the shape and color are not quite the same for the two men, this raises a difficulty, hastily solved, or rather covered up, by declaring what each sees to be purely "subjective"—though it would puzzle those who use this glib word to say what they mean by it. The truth seems to be that space—and time also—is much more complicated than it would appear to be from the finished structure of physics, and that the one all-embracing three-dimensional space is a logical construction, obtained by means of correlations from a crude space of six dimensions. The particulars occupying this six-dimensional space, classified in one way, form "things,"

from which with certain further manipulations we can obtain what physics can regard as matter; classified in another way, they form "perspectives" and "biographies," which may, if a suitable percipient happens to exist, form respectively the sense-data of a momentary or of a total experience. It is only when physical "things" have been dissected into series of classes of particulars, as we have done, that the conflict between the point of view of physics and the point of view of psychology can be overcome. This conflict, if what has been said is not mistaken, flows from different methods of classification, and vanishes as soon as its source is discovered.

In favor of the theory which I have briefly outlined, I do not claim that it is *certainly* true. Apart from the likelihood of mistakes, much of it is avowedly hypothetical. What I do claim for the theory is that it *may* be true, and that this is more than can be said for any other theory except the closely analogous theory of Leibniz. The difficulties besetting realism, the confusions obstructing any philosophical account of physics, the dilemma resulting from discrediting sense-data, which yet remain the sole source of our knowledge of the outer world—all these are avoided by the theory which I advocate. This does not prove the theory to be true, since probably many other theories might be invented which would have the same merits. But it does prove that the theory has a better chance of being true than any of its present competitors, and it suggests that what can be known with certainty is likely to be discoverable by taking our theory as a starting-point, and gradually freeing it from all such assumptions as seem irrelevant, unnecessary, or unfounded. On these grounds, I recommend it to attention as a hypothesis and a basis for further work, though not as itself a finished or adequate solution of the problem with which it deals.

CAMBRIDGE, ENGLAND. BERTRAND RUSSELL.

NEWTON'S HYPOTHESES OF ETHER AND OF GRAVITATION FROM 1693 TO 1726.

AFTER his correspondence with Bentley and a letter of 1693 to Leibniz, referred to in §VI below, Newton does not seem to have made any more pronouncements about the ether or the nature of gravitation until the Latin edition of his *Opticks,* which was published in 1706. Discussions[1] on the cause of gravitation were greatly stimulated by the appearance of Newton's *Principia* in 1687, and continued — especially between Leibniz and Huygens — until 1694; but it would not be relevant to our present subject to consider these discussions further here.

We have seen that Newton himself did not regard gravitation as an essential property of matter, implanted in it by the Creator. In the *Principia,* he merely considered it as a starting-point for mathematical deductions; the discovery of its nature and cause was a physical and not a mathematical problem. From the letters of 1692 and 1693 to Bentley we learn that he did not pretend to know what the cause of gravity might be, but it seemed to him incomprehensible that matter should act on other matter without the intervention of a medium. Still he explicitly stated that he did not express any opinion as to "whether this agent be material or immaterial." In the first edition of the *Opticks* (1704) there was no mention in the "Queries" of the ether or gravitation.

[1] Rosenberger, *Isaac Newton und seine physikalischen Principien,* pp. 227-248.

I.

In the preface to Newton's *Opticks: or, a Treatise of the Reflexions, Refractions, Inflexions, and Colours of Light,* which was published at London in 1704, he mentioned that the work consisted of: (1) The optical memoir of 1675; (2) Additions made in 1687; (3) Stray papers collected after this date into the third Book and the last proposition of the second Book. The printing of the work was delayed so that controversy might be avoided,[2] and even then it was only the importunity of his friends that led him to print at all. Newton's optical memoir of 1672 is not mentioned, and it is just possible that Newton's words: "If any papers writ on this subject are got out of my hands, they are imperfect and were perhaps written before I had tried all the experiments here set down, and fully satisfied myself about the laws of colors," refer to these memoirs.

The *Opticks* of 1704[3] is only hypothetical—at least intentionally[4]—in the "Queries" at the end. Of these[5] the first English edition only contained the first seven, the first sentence of the eighth, the ninth, the first two-thirds of the tenth, the first third of the eleventh, the twelfth to the fifteenth inclusive, and part of the sixteenth. The first Latin edition of 1706[6] contained in addition the second part of the eighth, the last third of the tenth, the last two-thirds of the eleventh, and the twenty-fifth to the thirty-first inclusive. The second English edition of 1717[7] contained in

[2] Hooke died on March 3, 1703.

[3] German annotated translation by William Abendroth in Nos. 96 and 97 of *Ostwald's Klassiker* (Leipsic, 1898). An account of the contents of the *Opticks* is given by Rosenberger on pp. 294-301 of his book; the "Queries" were dealt with on pp. 301-327.

[4] See below, toward the end of the present section.

[5] Cf. Rosenberger, *op. cit.*, pp. 302-303.

[6] *Optice: sive de Reflexionibus, Refractionibus, Inflexionibus et Coloribus Lucis Libri Tres*; translated by Samuel Clarke, London, 1706. Other editions of this Latin translation were published in 1719, 1740, 1747, 1749, and 1773.

[7] *Opticks: or, a Treatise of the Reflections, Refractions, Inflections and Colours of Light.* The Second Edition, with Additions. London, 1717.

addition the last third of the eighth, the seventeenth to the twenty-fourth inclusive, and many additions to the thirty-first. Other editions (1718, 1721, 1730) published in Newton's lifetime or revised by Newton before his death contained various additions to and omissions from many of the queries, especially the last.[8]

Rosenberger[9] has remarked that "while in the *Opticks* of 1704 the observations and measurements are taken without any alteration, on account of their unsurpassed accuracy, from the great optical memoir of 1675, the theory, or rather the hypothesis which is its foundation, receives a completely new form. In the memoir of 1675, Newton still considered it likely that the rays of light excite vibrations in the ether contained in bodies when the rays meet this ether, and that these vibrations allow the transmission of light or cause its reflection according to the phase in which they meet the rays. In the *Opticks,* on the other hand, after he had had to let the ether drop, Newton considered rays of light without reference to any vibrations there may be in bodies and ascribed the alternating transmission and reflection of light which the colored rings indicate, merely to the internal nature of rays of light."

In fact, Newton began the book with the words: "My design in this book is not to explain the properties of light by hypotheses, but to propose and prove them by reason and experiments: in order to which I shall premise the following definitions and axioms."

In spite of this declaration, Newton was obliged, as Rosenberger[10] has pointed out, to attribute hypothetical properties to rays of light,—that the rays have "fits of easy transmission and reflection." But an addition (Query 17) to the second edition of 1717 explains these "fits" by

[8] This is practically what Rosenberger says; but the English edition of 1721 is unaltered from the second edition. The second edition is reprinted in Vol. IV of *Horsley*.

[9] *Op. cit.,* pp. 293-294.

[10] *Op. cit.,* p. 296.

the hypothesis of undulations which are excited in bodies by the rays of light.

With regard to Rosenberger's view that Newton was gradually forced to give up the assumption of an ether, this, as we shall see, is by no means likely if we consider Newton's own words without prejudice. Rosenberger, indeed, seems much too prone to attribute to Newton himself the views which some of his school held.

In the next section we will review those queries added to the edition of 1706 which have a bearing on our present subject. In the third section, we must consider the second edition (1713) of the *Principia*. In the fourth section we consider the relevant queries added to the edition of the *Opticks* published in 1717. In the fifth section reference is made to Newton's later (1716, 1725) remarks on Leibniz's criticism that he viewed gravitation as an "occult cause" and a "miracle"; and the sixth section deals with Newton's opinions on an ether and the nature of gravitation from the general point of view which has been reached.

II.

The twenty-seventh query is[11]

"Are not all hypotheses erroneous which have hitherto been invented for explaining the phenomena of light by new modifications of the rays? For those phenomena depend not upon new modifications, as has been supposed, but upon the original and unchangeable properties of the rays."

The long twenty-eighth query[12] begins:

"Are not all hypotheses erroneous in which light is supposed to consist in pression or motion propagated through a fluid medium? For in all these hypotheses the phenomena of light have been hitherto explained by supposing that

[11] *Horsley,* Vol. IV, pp. 231-232. Cf. Rosenberger, *op. cit.*, p. 315.
[12] *Horsley,* Vol. IV, pp. 232-238. Cf. Rosenberger, *op. cit.*, pp. 315-318.

they arise from new modifications of the rays, which is an erroneous supposition.

"If light consisted only in pression propagated without actual motion, it would not be able to agitate and heat the bodies which refract and reflect it. If it consisted in motion propagated to all distances in an instant, it would require an infinite force every moment in every shining particle to generate that motion. And if it consisted in pression or motion propagated either in an instant or in time, it would bend into the shadow. For pression or motion cannot be propagated in a fluid in right lines beyond an obstacle which stops part of the motion, but will bend and spread every way into the quiescent medium which lies beyond the obstacle.[13] Gravity tends downwards, but the pressure of water arising from gravity tends every way with as much force sideways as downwards and through crooked passages as through straight ones. The waves on the surface of stagnating water, passing by the sides of a broad obstacle which stops part of them, bend afterwards, and dilate themselves gradually into the quiet water behind the obstacle. The waves, pulses or vibrations of the air wherein sounds consist bend manifestly; though not so much as the waves of water. For a bell or a cannon may be heard beyond a hill which intercepts the sight of the sounding body, and sounds are propagated as readily through crooked pipes as through straight ones. But light is never known to follow crooked passages nor to bend into the shadow. For the fixed stars, by the interposition of any of the planets, cease to be seen. And so do the parts of the sun by the interposition of the moon, Mercury or Venus. The rays which pass very near to the edges of any body are bent a little by the action of the body, as we showed above; but this bending is not toward but from the shadow, and is performed only in the passage of the ray

[13] *Principia*, Book II, Prop. XLII.

by the body and at a very small distance from it. So soon as the ray is past the body it goes right on."

In the course of this query, Newton remarked:[14] "And against filling the heavens with fluid mediums, unless they be exceeding rare, a great objection arises from the regular and very lasting motions of the planets and comets in all manner of courses through the heavens. For thence it is manifest that the heavens are void of all sensible resistance and, by consequence, of all sensible matter."

Another part of the same query is:[15]

"And therefore to make way for the regular and lasting motions of the planets and comets it is necessary to empty the heavens of all matter, except perhaps some very thin vapors, steams or effluvia arising from the atmosphere of the earth, planets and comets, and of such an exceedingly rare ethereal medium as we described above. A dense fluid can be of no use for explaining the phenomena of nature, the motions of the planets and comets being better explained without it. It serves only to disturb and retard the motions of those great bodies and make the frame of nature languish. And in the pores of bodies it serves only to stop the vibrating motions of their parts wherein their heat and activity consists. And as it is of no use and hinders the operations of nature and makes her languish, so there is no evidence for its existence, and therefore it ought to be rejected. And if it be rejected, the hypotheses that light consists in pression or motion propagated through such a medium are rejected with it.

"And for rejecting such a medium we have the authority of the oldest and most celebrated philosophers of Greece and Phoenicia, who made a vacuum and atoms and the gravity of atoms the first principles of their philosophy, tacitly attributing gravity to some other cause than dense matter.

[14] *Horsley,* Vol. IV, p. 234.
[15] *Horsley,* Vol. IV, pp. 236-238, Rosenberger, *op. cit.,* pp. 343-344.

Later philosophers banish the consideration of such a cause out of natural philosophy, feigning hypotheses for explaining all things mechanically and referring other causes to metaphysics: whereas the main business of natural philosophy is to argue from phenomena without feigning hypotheses and to deduce causes from effects till we come to the very first cause, which certainly is not mechanical; and not only to unfold the mechanism of the world, but chiefly to resolve these and such like questions. What is there in places almost empty of matter and whence is it that the sun and planets gravitate toward one another without dense matter between them? Whence is it that nature does nothing in vain; and whence arises all that order and beauty which we see in the world? To what end are comets, and whence is it that planets move all one and the same way in concentric orbits, while comets move all manner of ways in orbits very eccentric, and what hinders the fixed stars from falling upon one another? Does it not appear from phenomena that there is a Being incorporeal, living, intelligent, omnipresent, who in infinite space, as it were in his sensory, sees the things themselves intimately?" and so on.

In the twenty-ninth query, Newton[16] proceeded to develop his own emission-theory of light. It begins:

"Are not the rays of light very small bodies emitted from shining substances? For such bodies will pass through uniform mediums in right lines without bending into the shadow, which is the nature of the rays of light. They will also be capable of several properties and be able to conserve their properties unchanged in passing through several mediums, which is another condition of the rays of light."

"It is remarkable," said Rosenberger,[17] "that the state-

[16] *Horsley*, Vol. IV, pp. 238-241. Cf. Rosenberger, *op. cit.*, pp. 319-321.
[17] *Op. cit.*, pp. 320-321.

ment [in this query] of the necessity of attractive forces and [in the preceding one] of the impossibility of an ether is followed by a reference intercalated in the edition of 1717, to the derivation of this attraction by means of this ether. This reference is: 'What I mean in this question by a *vacuum* and by the attraction of the rays of light towards glass or crystal may be understood by what was said in the 18th, 19th, and 20th questions.'[18] Apparently Newton wished to show, even in his queries, that his theories are reconcilable with all hypotheses...."

It is obviously incorrect to speak, as Rosenberger does, of Newton's "statement...of the impossibility of an ether." As a rule, we have no means of deciding, from the "Queries," what Newton's real opinions on an ether were; but the letters to Bentley throw a great deal of light on the subject. Indeed, the weight of evidence seems to tell against Rosenberger's[19] view and in favor of the view which has become traditional, that Newton believed in some sort—not the Cartesian—of ether on the ground that he could not imagine a propagation of force without a medium.[20]

The thirtieth query[21] begins: "Are not gross bodies and light convertible into one another; and may not bodies receive much of their activity from the particles of light which enter their composition?" We know that for long after this, the matter of light was considered as a possible part of the subject-matter of chemistry.

The thirty-first query[22] was greatly added to in later editions, but we will here collect all that is said in it in its final form about the subjects that interest us at present. It begins:

"Have not the small particles of bodies certain powers,

[18] *Horsley,* Vol. IV, p. 241. These queries are given in § IV below.
[19] Cf. also *op. cit.,* pp. 333-334.
[20] Cf. § VI below.
[21] *Horsley,* Vol. IV, pp. 241-242. Cf. Rosenberger, *op. cit.,* pp. 321-322.
[22] *Horsley,* Vol. IV, pp. 242-264. Cf. Rosenberger, *op. cit.,* pp. 322-327.

virtues or forces by which they act at a distance, not only upon the rays of light for reflecting, refracting and inflecting them, but also upon one another for producing a great part of the phenomena of nature? For it is well known that bodies act one upon another by the attractions of gravity, magnetism and electricity, and these instances show the tenor and course of nature and make it not improbable that there may be more attractive powers than these. For nature is very consonant and conformable to herself. How these attractions may be performed, I do not here consider. What I call attraction may be performed by impulse or by some other means unknown to me. I use that word here to signify only in general any force by which bodies tend toward one another, whatever be the cause. For we must learn from the phenomena of nature what bodies attract one another and what are the laws and properties of the attraction, before we inquire the cause by which the attraction is performed. The attractions of gravity, magnetism and electricity reach to very sensible distances and so have been observed by vulgar eyes, and there may be others which reach to so small distances as hitherto escape observation; and perhaps electrical attraction may reach to such small distances even without being excited by friction."

"All bodies," said Newton,[23] "seem to be composed of hard particles." And again:[24] ". . . . it seems probable to me that God in the beginning formed matter in solid, massy, hard, impenetrable, movable particles of such sizes and figures and with such other properties and in such proportion to space[25] as most conduced to the end for

[23] *Horsley,* Vol IV, p. 251. [24] *Ibid.*, pp 260-262.

[25] "*Eoque numero et quantitate pro ratione spatii in quo futurum erat ut moverentur.* Dr. Clarke's first Latin; and so the sentence stands in Dr. Clarke's second Latin edition, which in most places was so corrected as to agree exactly with the second English. And to make sense of the passage, something is evidently wanting here to answer to the words of the Latin, *in quo futurum erat ut moverentur.* For to speak of particles of matter as bearing proportion to space indefinitely were absurd."—Note by Horsley.

which he formed them, and that these primitive particles being solids, are incomparably harder than any porous bodies compounded of them; even so very hard as never to wear or break in pieces; no ordinary power being able to divide what God himself made one in the first creation. While the particles continue entire, they may compose bodies of one and the same nature and texture in all ages; but should they wear away or break in pieces, the nature of things depending on them would be changed. Water and earth composed of old worn particles and fragments of particles would not be of the same nature and texture now with water and earth composed of entire particles in the beginning. And therefore that nature may be lasting, the changes of corporeal things are to be placed only in the various separations and new associations and motions of these permanent particles; compound bodies being apt to break, not in the midst of solid particles but where those particles are laid together and only touch in a few points.

"It seems to me further that these particles have not only a *vis inertiae* accompanied with such passive laws of motion as naturally result from that force, but also that they are moved by certain active principles, such as is that of gravity and that which causes fermentation and the cohesion of bodies. These principles I consider not as occult qualities, supposed to result from the specific forms of things, but as general laws of nature by which the things themselves are formed: their truth appearing to us by phenomena, though their causes be not yet discovered. For these are manifest qualities, and their causes only are occult. And the Aristotelians gave the name of occult qualities not to manifest qualities but to such qualities only as they supposed to lie hid in bodies and to be the unknown causes of manifest effects: such as would be the causes of gravity and of magnetic and electric attractions and of

fermentations, if we should suppose that these forces or actions arose from qualities unknown to us and incapable of being discovered and made manifest. Such occult qualities put a stop to the improvement of natural philosophy, and therefore of late years have been rejected. To tell us that every species of things is endowed with an occult specific quality by which it acts and produces manifest effects, is to tell us nothing: but to derive two or three general principles of motion from phenomena, and afterwards to tell us how the properties of motion follow from phenomena, and afterwards to tell us how the properties and actions of all corporeal things follow from those manifest principles, would be a very great step in philosophy, though the causes of those principles were not yet discovered. And therefore I scruple not to propose the principles of motion above mentioned, they being of very general extent, and leave their causes to be found out.

"Now by the help of these principles, all material things seem to have been composed of the hard and solid particles above mentioned, variously associated in the first creation by the counsel of an intelligent Agent. For it became Him who created them to set them in order. And if he did so, it is unphilosophical to seek for any other origin of the world or to pretend that it might arise out of a chaos by the mere laws of nature; though being once formed, it may continue by those laws for many ages. For while comets move in very eccentric orbits in all manner of positions, blind Fate could never make all the planets move one and the same way in orbits concentric, some inconsiderable irregularities excepted, which may have arisen from the mutual actions of comets and planets upon one another, and which will be apt to increase till this system wants a reformation. Such a wonderful uniformity in the planetary system must be allowed the effect of choice. And so must the uniformity in the bodies of animals."

III.

Between the publication of the Latin translation and the second edition of the *Opticks* came that of the second edition of the *Principia* in 1713, which was very carefully edited by Roger Cotes.[26] Frequent mention of this edition has been already made in these articles:[27] what is of the greatest interest to us in this connection is that Newton, while emphasizing his neutral attitude toward the question as to whether *gravitation* was an essential property of bodies or no, added the famous *regulae philosophandi.* In the third rule, extension, hardness, impenetrability, mobility *and inertia* are classed together as universal qualities of bodies which can neither be increased nor diminished. In the second place, Cotes's preface, which expressed the views of Newton's school rather than of Newton himself, is usually supposed, and somewhat incorrectly supposed, to contain a distinct enunciation of the view—which Newton repeatedly disclaimed[28]—that gravity is an essential property of bodies.

Newton's own changes relating to our present subject in this edition have already been dealt with in §§ III and IV of my article in the April number of this magazine, and chiefly emphasize Newton's aversion to being supposed to state that gravity was an essential property of bodies. We will now fix our attention on Cotes's preface. In it is the passage:[29]

"Since, then, all bodies, whether upon earth or in the heavens, are heavy, so far as we can make any experiments or observations concerning them, we must certainly allow

[26] See on this point, Edleston, *op. cit.*; Brewster, *op. cit.*, Vol. I, pp. 312-318; Vol. II, pp. 248-252, 254; Rouse Ball, *op. cit.*, pp. 124-135; Rosenberger, *op. cit.*, pp. 368-385.

[27] See *Monist* for October, 1914, § XV, and for April, 1915, § IV.

[28] It should be noticed that Newton expressly declined to see all of Cotes's preface lest he should be considered to be responsible for it (cf. Rosenberger, *op. cit.*, p. 373).

[29] Cf. Rosenberger, *op. cit.*, pp. 376-377.

that gravity is found in all bodies universally. And just as we ought not to suppose that any bodies can be otherwise than extended, movable or impenetrable; so we ought not to conceive that any bodies can be otherwise than heavy. The extension, mobility and impenetrability of bodies become known to us only by experiments; and in the very same manner their gravity becomes known to us. All bodies we can make any observations upon are extended, movable and impenetrable; and thence we conclude that all bodies, even those concerning which we have no observations, are extended and movable and impenetrable. So we find that all bodies on which we can make observations are heavy; and thence we conclude that all bodies, even those of which we have no observations, are heavy also. If any one should say that the bodies of the fixed stars are not heavy because their gravity is not yet observed, he may say for the same reason that they are neither extended nor movable nor impenetrable, because these affections of the fixed stars are not yet observed. In short, either gravity must have a place among the primary qualities of all bodies, or extension, mobility and impenetrability must not. And if the nature of things is not rightly explained by the gravity of bodies, it will not be rightly explained by their extension, mobility and impenetrability.

"Some I know disapprove this conclusion, and mutter something about occult qualities.[30] They continually are caviling with us that gravity is an occult property and occult causes are to be quite banished from philosophy. But to this the answer is easy, that those are indeed occult causes whose existence is occult and imagined but not proved, but not those whose real existence is clearly demonstrated by observations. Therefore gravity can by no

[30] Leibniz in 1710 had accused Newton of introducing occult qualities and miracles into philosophy. On the subsequent controversy between Leibniz, Clarke, and others, see Brewster, *op. cit.*, Vol. II, pp. 282-289; Rosenberger, *op. cit.*, pp. 512-519.

means be called an occult cause of the celestial motions; because it is plain from the phenomena that such a virtue does really exist. Those rather have recourse to occult causes who postulate imaginary vortices of a matter entirely fictitious and imperceptible by our senses to direct those motions. But shall gravity be therefore called an occult cause and thrown out of philosophy because the cause of gravity is occult and not yet discovered? Those who affirm this should be careful not to fall into an absurdity that may overturn the foundations of all philosophy. For causes are wont to proceed in a continuous chain from those that are more compounded to those that are more simple; and when we have arrived at the most simple cause, we can go no farther. Therefore no mechanical account or explanation of the most simple cause is to be expected or given; for if it could be given, the cause were not the most simple. These most simple causes will you then call occult and reject them? Then you must reject those that immediately depend upon them, and those which depend upon the last, till philosophy is quite cleared and disencumbered of all causes.

"Some there are who say that gravity is preternatural and call it a perpetual miracle. Therefore they would have it rejected because preternatural causes have no place in physics. It is hardly worth while to spend time in answering this ridiculous objection, which overturns all philosophy. For either they will deny gravity to be in bodies, which cannot be said, or else they will call it preternatural because it is not produced by the other affections of bodies and therefore not by mechanical causes. But certainly there are primary affections of bodies, and these, because they are primary, have no dependence on the others. Let them consider whether all these are not in like manner preternatural, and in like manner to be rejected, and then what kind of philosophy we are like to have."

With this passage it is important to compare the following letter written by Cotes to Samuel Clarke. It is all the more important as it does not seem to be sufficiently emphasized by Rosenberger.[31]

Cotes wrote to Samuel Clarke[32] on June 25, 1713:[33]

"I received your very kind letter. I return you my thanks for your corrections of the Preface, and particularly for your advice in relation to that place where I seemed to assert gravity to be essential to bodies. I am fully of your mind that it would have furnished matter for caviling, and therefore I struck it out immediately upon Dr. Cannon's mentioning your objection to me, and so it never was printed. The impression of the whole book was finished about a week ago.

"My design in that passage was not to assert gravity to be essential to matter, but rather to assert that we are ignorant of the essential properties of matter, and that, in respect of our knowledge, gravity might possibly lay as fair a claim to that title as the other properties which I mentioned. For I understand by essential properties such properties without which no others belonging to the same substance can exist: and I would not undertake to prove that it were impossible for any of the other properties of bodies to exist without even extension."

The second edition of the *Principia* seems to have been published at the beginning of July, 1713. Edleston[34] thus commented on the above letter:

"It appears from the above letter that a meaning has been given to expressions in Cotes's Preface which he did not intend them to convey. He has been understood to assert that gravity is an essential property of bodies: his words are: '*Inter primarias qualitates corporum universorum vel*

[31] *Op. cit.*, p. 374. But cf. pp. 381-382.
[32] On Clarke see Rosenberger, *op. cit.*, pp. 287-288.
[33] Edleston, *op. cit.*, pp. 158-159. [34] *Ibid.*, pp. 159-160.

Gravitas habebit locum; vel Extensio, Mobilitas et Impenetrabilitas non habebunt.'" This passage is translated above.

IV.

The first notable change in the second English edition of 1717 was a second "Advertisement" dated July 16, 1717:

"And, to show that I do not take gravity for an essential property of bodies, I have added one question concerning its cause, choosing to propose it by way of a question because I am not yet satisfied about it for want of experiments."[35]

Queries eighteen to twenty are as follows:[36]

"Qu. 18. If in two large tall inverted cylindrical vessels of glass two little thermometers be suspended so as not to touch the vessels, the air be drawn out of one of these vessels, and these vessels thus prepared be carried out of a cold place into a warm one; the thermometer *in vacuo* will grow warm as much and almost as soon as the thermometer which is not *in vacuo*. And when the vessel is carried back into the cold place, the thermometer *in vacuo* will grow cold almost as soon as the other thermometer. Is not the heat of the warm room conveyed through the vacuum by the vibrations of a much subtler medium than air, which, after the air was drawn out, remained in the vacuum? And is not this medium the same as that medium by which light is refracted or reflected, and by whose vibrations light communicates heat to bodies, and is put into fits of easy reflection and easy transmission? And do not the vibrations of this medium in hot bodies contribute to the intenseness and duration of their heat? And do not hot

[35] This declaration was probably drawn from Newton by the then recent controversy between Leibniz and Clarke. Cf. also *Macc. Corr.*, Vol. II, p.437. At the end of the preface to the second edition of the *Opticks* was added to the preface to the first edition the date: April 1, 1704. There is a similar peculiarity about the preface to the *Principia*; and Edleston (*op. cit.*, p. lxxi) remarked that the dispute with Leibniz about the invention of the calculus had probably taught Newton the importance of dates.

[36] *Horsley*, Vol. IV, pp. 223-224. Cf. Rosenberger, *op. cit.*, pp. 306-307.

bodies communicate their heat to contiguous cold ones by the vibrations of this medium propagated from them into the cold ones? And is not this medium exceedingly more rare and subtle than the air and exceedingly more elastic and active? And does it not readily pervade all bodies? And is it not (by its elastic force) expanded through all the heavens?

"Qu. 19. Does not the refraction of light proceed from the different density of this ethereal medium in different places, the light receding always from the denser parts of the mediums? And is not the density thereof greater in free and open spaces void of air and other grosser bodies than within the pores of water, glass, crystal, gems, and other compact bodies? For when light passes through glass or crystal, and, falling very obliquely upon the farther surface thereof, is totally reflected, the total reflection ought to proceed rather from the density and vigor of the medium without and beyond the glass than from the rarity and weakness thereof.

"Qu. 20. Does not this ethereal medium, in passing out of water, glass, crystal and other compact and dense bodies into empty spaces, grow denser and denser by degrees, and by that means refract the rays of light not in a point but by bending them gradually in curve lines? And does not the gradual condensation of this medium extend to some distance from the bodies, and thereby cause the inflections of the rays of light which pass by the edges of dense bodies at some distance from the bodies?"

The twenty-first query runs:[37]

"Qu. 21. Is not this medium much rarer within the dense bodies of the sun, stars, planets, and comets than in the empty celestial spaces between them? And in passing from them to great distances, does it not grow denser and

[37] *Opticks*, 1721, pp. 325-327; *Horsley*, Vol. IV, pp. 224-225. Cf. Rosenberger, *op. cit.*, pp. 307-308.

denser perpetually and thereby cause the gravity of those great bodies toward one another and of their parts toward the bodies, every body endeavoring to go from the denser parts of the medium toward the rarer? For if this medium be rarer within the sun's body than at its surface, and rarer there than at the hundredth part of an inch from its body, and rarer there than at the fiftieth part of an inch from its body, and rarer there than at the orb of Saturn; I see no reason why the increase of density should stop anywhere and not rather be continued through all distances from the Sun to Saturn and beyond. And though this increase of density may at great distances be exceeding slow, yet if the elastic force of this medium be exceeding great it may suffice to impel bodies from the denser parts of the medium towards the rarer with all that power which we call gravity. And that the elastic force of this medium is exceeding great may be gathered from the swiftness of its vibrations. Sounds move about 1140 English feet in a second of time, and in seven or eight minutes of time they move about one hundred English miles. Light moves from the sun to us in about seven or eight minutes of time, which distance is about 70,000,000 English miles, supposing the horizontal parallax of the sun to be about 12 seconds. And the vibrations of pulses of this medium, that they may cause the alternate fits of easy transmission and easy reflection, must be swifter than light, and by consequence above 700,000 times swifter than sounds. And therefore the elastic force of this medium in proportion to its density must be above 700,000 times 700,000 (that is, above 490,000,000,000) times greater than the elastic force of the air is in proportion to its density. For the velocities of the pulses of elastic mediums are in a subduplicate ratio of the elasticities and the rarities of the mediums taken together.

"As attraction is stronger in small magnets than in great ones in proportion to their bulk, and gravity is

greater in the surfaces of small planets than in those of great ones in proportion to their bulk, and small bodies are agitated much more by electric attraction than great ones, so the smallness of the rays of light may contribute very much to the power of the agent by which they are refracted. And so if any one should suppose that ether (like our air) may contain particles which endeavor to recede from one another (for I do not know what this ether is) and that its particles are exceedingly smaller than those of air or even than those of light, the exceeding smallness of its particles may contribute to the greatness of the force by which those particles may recede from one another, and thereby make that medium exceedingly more rare and elastic than air and by consequence exceedingly less able to resist the motions of projectiles and exceedingly more able to press upon gross bodies by endeavoring to expand itself."

The twenty-second query is:[38]

"May not planets and comets and all gross bodies perform their motions more freely and with less resistance in this ethereal medium than in any fluid which fills all space adequately without leaving any pores, and by consequence is much denser than quicksilver or gold. And may not its resistance be so small as to be inconsiderable? For instance, if this ether (for so I call it) should be supposed 700,000 times more elastic than our air and above 700,000 times more rare, its resistance would be above 600,000,000 times less than that of water. And so small a resistance would scarce make any sensible alteration in the motions of the planets in ten thousand years. If any one would ask how a medium can be so rare, let him tell me how the air in the upper parts of the atmosphere can be above hundred thousand times rarer than gold. Let him also tell me how an electric body can by friction emit an exhalation so rare

[38] *Horsley*, Vol. IV, pp. 225-226; Rosenberger, *op. cit.*, pp. 308-309.

and subtle and yet so potent as by its emission to cause no sensible diminution of the weight of the electric body, and to be expanded through a sphere whose diameter is above two feet and yet to be able to agitate and carry up leaf-copper or leaf-gold at the distance of above a foot from the electric body? And how the effluvia of a magnet can be so rare and subtle as to pass through a plate of glass without any resistance or diminution of their force, and yet so potent as to turn a magnetic needle beyond the glass?"

To base on the eighteenth query to the twenty-second query an opinion that as time went on Newton inclined more and more to the undulatory theory of light with its necessary assumption of an ether, and here, in 1717, denied the doctrine of his school which asserted action at a distance is, according to Rosenberger,[39] incorrect. It seems true that Newton, for what were then good reasons, never abandoned the emission theory. It is also true that the hint as to the derivation of gravitation from different densities of the ether is merely vague. But this does not prove that Newton had not a decided leaning toward the notion that action was always propagated by a medium. But Rosenberger makes a very apposite remark when he states that the matter of these "Queries" is taken from the "Hypothesis" of 1675 and the letter to Boyle of 1679. It may be that these old speculations were again brought forward to show that Newton had studied theories of the ether as carefully as his opponents—Leibniz and the Cartesians.[40]

Finally, an addition to the twenty-ninth query is referred to in § II above.

V.

In a letter of February 26, 1716,[41] to the Abbé Conti,

[39] *Op. cit.*, pp. 309-311. [40] *Ibid.*, p. 311.

[41] *Horsley*, Vol. IV, p. 597; Brewster, *op. cit.*, Vol. II, pp. 288-289; Rosenberger, *op. cit.*, pp. 382-383.

Newton commented on Leibniz's use,[42] in 1710, of the terms "occult causes" and "miracles"; but Newton's remarks are not of much importance. This was[43] also dwelt upon in the unsigned preface to the edition of the *Commercium Epistolicum* published in 1725, and this preface is known[44] to have been written by Newton himself.

Finally, in the third edition of the *Principia*, which was edited by Henry Pemberton and published in 1726,[45] there is, as we saw in § IV of my article in *The Monist* for April, 1915, one addition emphasizing Newton's wish not to be understood as asserting that gravity was an essential property of bodies.

VI.

Rosenberger[46] says: "It results quite clearly from Newton's utterances that he held that a kinetic or even a physical explanation of gravity as an effect brought about by an intervening medium is impossible. In the *Principia* he decisively refuted the possibility of an explanation of gravity and the celestial motions by Descartes's vortices. But he also refuted the many attempts at the improvement of this doctrine which were given at that time by destroying their fundamental supposition—that there exists an ethereal matter which fills all space—by his researches on the resistance which this matter must present to the celestial motions. It might have been urged that by this the impossibility of an ether in general was not proved: only of one whose density could be measured by us. It might have been urged that Newton himself admitted a light-matter which is distributed everywhere like light and must thus just like the ether hinder the motions of the planets. But Newton explained that his corpuscles

[42] See above, § III.

[43] Rosenberger, *op. cit.*, pp. 383-384.

[44] Brewster, *op. cit.*, Vol. II, p. 75.

[45] Cf. Brewster, *op. cit.*, Vol. II, pp. 378-384, 549-556; Rouse Ball, *op. cit.*, pp. 135-136.

[46] *Op. cit.*, p. 409.

are much too thinly distributed in space for their resistance to be sensible, while the Cartesian ether, which completely fills space, must under all circumstances give rise to a very great resistance."

Newton's views on the subjects which concern us at present are thus summarized by Rosenberger.[47]

"All particles of matter accessible to our senses and all bodies move as if they attracted one another in the direct ratio of their masses and the inverse ratio of the squares of their distances from one another. This mutual action is not action at a distance but is brought about by some intermediary not yet certainly to be determined. Since, however, a matter suitable to play the part of this intermediary cannot be found, this agent is probably immaterial and in all likelihood God himself. It is doubtful whether God does this directly or indirectly, though probably the former. Hence follows that the mutual action in all matter is not an essential and necessary property of it like impenetrability and mobility, but a quality given to matter by the creator in perfect freedom and in the way which seemed proper to him."

It is not impossible that this was Newton's real view. But it seems impossible to doubt that Newton decidedly leaned toward the hypothesis of a very rare ether. Certainly he thought that he had refuted the Cartesian theory of a dense ether. And Newton himself has given some further indication of what his views were, and this indication seems, like the letters to Bentley, to support our conjecture. Leibniz had asked Newton what was his opinion of what Huygens had said in the appendix, *Discours de la Cause de la Pesanteur,* to his *Traité de la Lumière* of 1690. On October 26, 1693, Newton replied:[48]

"I cannot admit that a subtle matter fills the heavens,

[47] *Op. cit.,* p. 422.
[48] Mentioned in another connection by Rosenberger, *op. cit.,* pp. 461-462.

for the celestial motions are too regular to arise from vortices, and vortices would only disturb the motion. But if any one should explain gravity and all its laws by the action of some subtle mediums, and should show that the motions of the planets and comets were not disturbed by this matter, I should by no means oppose it."

It would seem that Newton felt just as strongly as did Descartes or Huygens or Leibniz or Faraday or Maxwell a need for a medium by which to transmit force. A weapon sometimes used against Cartesian doctrines by those whose interests or emotions led them to condemn what they fancied might be construed as a slight on the omnipotence of God, was that Descartes's world was so planned as to leave God out of account altogether. But something like this is the ideal of every scientific man. However pious a man of science may be—and some even eminent men of science have been models of unthinking devotion to certain religious sects — he will try to explain by natural causes phenomena which have hitherto appeared inexplicable except as miracles wrought by God. It is difficult to imagine that religious belief could ever interfere with this postulate of scientific investigation, and thus it can hardly be believed that Newton preferred to help himself out by supposing certain actions on the part of God, when there was a chance that gravitation might be explained by the hypothesis of a rarefied ether.

PHILIP E. B. JOURDAIN.

CAMBRIDGE, ENGLAND.

CRITICISMS AND DISCUSSIONS.

THE WORKS OF WILLIAM OUGHTRED.*

CLAVIS MATHEMATICAE.

William Oughtred (1574 (?)–1660), though by profession a clergyman, was one of the world's great teachers of mathematics and should still be honored as the inventor of that indispensable mechanical instrument, the slide-rule. It is noteworthy that he showed a marked disinclination to give his writings to the press. His first paper on sun-dials was written at the age of twenty-three, but we are not aware that more than one brief mathematical manuscript was printed before his fifty-seventh year. In every instance, publication in printed form seems to have been due to pressure exerted by one or more of his patrons, pupils or friends. Some of his manuscripts were lent out to his pupils who prepared copies for their own use. In some instances they urged upon him the desirability of publication and assisted in preparing copy for the printer. The earliest and best known book of Oughtred was his *Clavis mathematicae.* As he himself informs us, he was employed by the Earl of Arundel about 1628 to instruct the Earl's son, Lord William Howard (afterwards Viscount Stafford), in the mathematics. For the use of this young man Oughtred composed a treatise on algebra which was published in Latin in the year 1631 at the urgent request of a kinsman of the young man, Charles Cavendish, a patron of learning.

The *Clavis mathematicae,*[1] in its first edition of 1631, was a booklet of only 88 small pages. Yet it contained in very condensed form the essentials of arithmetic and algebra as known at that time.

Aside from the addition of four tracts, the 1631 edition underwent some changes in the editions of 1647 and 1648 which two are much alike. The twenty chapters of 1631 are reduced to nineteen in 1647 and in all the later editions. Numerous minute alterations

* For further details see the author's article on "The Life of Oughtred" in *The Open Court,* August, 1915, where fuller references are given to some of the books cited here.

[1] The full title of the *Clavis* of 1631 is as follows: *Arithmeticae in numeris et speciebvs institvtio: Quae tvm logisticae, tvm analyticae, atqve adeo totivs*

from the 1631 edition occur in all parts of the books of 1647 and 1648. The material of the last three chapters of the 1631 edition is re-arranged with some slight additions here and there. The 1648 edition has no preface. In the print of 1652 there are only slight alterations from the 1648 edition; after that the book underwent hardly any changes, except for the number of tracts appended, and brief explanatory notes added at the close of the chapters in the English edition of 1694 and 1702. The 1652 and 1667 editions were seen through the press by John Wallis; the 1698 impression contains on the title-page the words: *Ex Recognitione D. Johannis Wallis, S.T.D. Geometriae Professoris Saviliani.*

The cost of publishing may be a matter of some interest. When arranging for the printing of the 1667 edition of the *Clavis,* Wallis wrote Collins:[2] "I told you in my last what price she [Mrs. Lichfield] expects for it, as I have formerly understood from her, viz., 40 l. for the impression, which is about 9½d. a book."

mathematicae, qvasi clavis est.—Ad nobilissimvm spectatissimumque invenem Dn. Guilelmvm Howard, Ordinis qui dicitur, Balnei Equitem, honoratissimi Dn. Thomae, Comitis Arvndeliae & Surriae, Comitis Mareschalli Angliae, &c filium—Londini, Apud Thomam Harpervm. M.DC.XXXI.

In all there appeared five Latin editions, the second in 1648 at London, the third in 1652 at Oxford, the fourth in 1667 at Oxford, the fifth in 1693 and 1698 at Oxford. There were two independent English editions: the first in 1647 at London, translated in greater part by Robert Wood of Lincoln College, Oxford, as is stated in the preface to the 1652 Latin edition; the second in 1694 and 1702 is a new translation, the preface being written and the book recommended by the astronomer Edmund Halley. The 1694 and 1702 impressions labored under the defect of many sense-disturbing errors due to careless reading of the proofs. All the editions of the *Clavis,* after the first edition, had one or more of the following tracts added on:

Eq. = *De Aequationum affectarum resolutione in numeris.*
Eu. = *Elementi decimi Euclidis declaratio.*
So. = *De Solidis regularibus tractatus.*
An. = *De Anatocismo, sive usura composita.*
Fa. = *Regula falsae positionis.*
Ar. = *Theorematum in libris Archimedis de Sphaera & cylindro declaratio.*
Ho. = *Horologia scioterica in plano, geometricè delineandi modus.*

The abbreviated titles given here are, of course, our own. The lists of tracts added to the *Clavis mathematicae* of 1631 in its later editions, given in the order in which the tracts appear in each edition, are as follows: *Clavis* of 1647, *Eq., An., Fa., Ho.; Clavis* of 1648, *Eq., An., Fa., Eu., So.; Clavis* of 1652, *Eq., Eu., So., An., Fa., Ar., Ho.; Clavis* of 1667, *Eq., Eu., So., An., Fa., Ar., Ho.*; *Clavis* of 1693 and 1698, *Eq., Eu., So., An., Fa., Ar., Ho.*; *Clavis* of 1694 and 1702, *Eq.*

The title-page of the *Clavis* was considerably modified after the first edition. Thus, the 1652 Latin edition has this title-page: *Guilelmi Oughtred Aetonensis, quondam Collegii Regalis in Cantabrigia Socii, Clavis mathematicae denvo limata, sive potius fabricata. Cum aliis quibusdam ejusdem commentationibus, quae in sequenti pagina recensentur. Editio tertia auctior & emendatior Oxoniae, Excudebat Leon. Lichfield, Veneunt apud Tho. Robinson. 1652.*

[2] Rigaud, *op. cit.,* Vol. II, p. 476.

As compared with other contemporary works on algebra, Oughtred's distinguishes itself for the amount of symbolism used, particularly in the treatment of geometric problems. Extraordinary emphasis was placed upon what he called in the *Clavis* the "analytical art."[3] By that term he did not mean our modern analysis or analytical geometry, but the art "in which by taking the thing sought as knowne, we finde out that we seeke."[4] He meant to express by it condensed processes of rigid, logical deduction expressed by appropriate symbols, as contrasted with mere description or elucidation by passages fraught with verbosity. In the preface to the first edition (1631) he says:

"In this little book I make known...the rules relating to fundamentals, collected together, just like a bundle, and adapted to the explanation of as many problems as possible."

As stated in this preface, one of his reasons for publishing the book, is "...that like Ariadne I might offer a thread to mathematical study by which the mysteries of this science might be revealed, and direction given to the best authors of antiquity, Euclid, Archimedes, the great geometrician Apollonius of Perga, and others, so as to be easily and thoroughly understood, their theorems being added, not only because to many they are the height and depth of mathematical science (I ignore the would-be mathematicians who occupy themselves only with the so-called practice, which is in reality mere juggler's tricks with instruments, the surface so to speak, pursued with a disregard of the great art, a contemptible picture), but also to show with what keenness they have penetrated, with what mass of equations, comparisons, reductions, conversions and disquisitions these heroes have ornamented, increased and invented this most beautiful science."

The *Clavis* opens with an explanation of the Hindu-Arabic notation of decimal fractions. Noteworthy is the absence of the words "million," "billion," etc. Although used on the continent by certain mathematical writers long before this, these words did not become current in English mathematical books until the eighteenth century. The author was a great admirer of decimal fractions, but failed to introduce the notation which in later centuries came to be

[3] See, for instance, the *Clavis mathematicae* of 1652, where he expresses himself thus (p. 11): "*Speciosa haec Arithmetica arti Analyticae (per quam ex sumptione quaesiti, tanquam noti, investigatur quaesitum) multo accommodatior est, quam illa numerosa.*"

[4] Oughtred, *The Key of the Mathematicks,* London, 1647, p. 4.

universally adopted. Oughtred wrote 0.56 in this manner $0\underline{|56;}$ the point he used to designate ratio. Thus 3:4 was written by him 3·4. The decimal point (or comma) was first used by the inventor of logarithms, John Napier, as early as 1616 and 1617. Although Oughtred had mastered the theory of logarithms soon after their publication in 1614 and was a great admirer of Napier, he preferred to use the dot for the designation of *ratio.* This notation of ratio is used in all his mathematical books, except in two instances. The two dots (:) occur as symbols of ratio in some parts of Oughtred's posthumous work, *Opuscula mathematica hactenus inedita,* Oxford, 1677, but may have been due to the editors and not to Oughtred himself. Then again the two dots (:) are used to designate ratio on the last two pages of the tables of the Latin edition of Oughtred's *Trigonometria* of 1657. In all other parts of that book the dot (.) is used. Probably some one who supervised the printing of the tables introduced the (:) on the last two pages, following the logarithmic tables, where methods of interpolation are explained. The probability of this conjecture is the stronger, because in the English edition of the *Trigonometrie,* brought out the same year (1657) but *after* the Latin edition, the notation (:) at the end of the book is replaced by the usual (.), except that in some copies of the English edition the explanations at the end are omitted altogether.

Oughtred introduces an interesting, and at the same time new, feature of an abbreviated multiplication and an abbreviated division of decimal fractions. On this point he took a position far in advance of his time. The part on abbreviated multiplication was re-written in slightly enlarged form and with some unimportant alterations in the later editions of the Clavis. We give it as it occurs in the revision. Four cases are given. In finding the product of $246\underline{|914}$ and $35\underline{|27,}$ "if you would have the Product without any Parts" (without any decimal part), "set the place of Unity of the lesser under the place of Unity in the greater: as in the Example," writing the figures of the lesser number in *inverse order.* From the example it will be seen that he begins by multiplying by 3, the right-hand digit of the multiplier. In the first edition of the *Clavis* he began with 7, the left digit. Observe also that he "carries" the nearest tens in the product of each lower digit and the upper digit one place to its

$$\begin{array}{cccccc}
2 & 4 & 6 & | \underline{9} & \underline{1} & \underline{4} \\
 & \underline{7} & \underline{2} & |5 & 3 & \\
\hline
7 & 4 & 0 & 7 & & \\
1 & 2 & 3 & 5 & & \\
 & & 4 & 9 & & \\
 & & 1 & 7 & & \\
\hline
8 & 7 & 0 & 8 & &
\end{array}$$

right. For instance, he takes $7 \times 4 = 28$ and carries 3, then he finds $7 \times 2 + 3 = 17$ and writes down 17.

The second case supposes that "you would have the Product with some places of parts" (decimals), say 4; "Set the place of Unity of the lesser Number under the Fourth place of the Parts of the greater." The multiplication of $246\underline{|914}$ by $35\underline{|27}$ is now performed thus:

$$
\begin{array}{cccccccc}
2 & 4 & 6 & \underline{|9} & \underline{1} & \underline{4} & & \\
 & & & & \underline{7} & \underline{2|} & 5 & 3 \\
\hline
7 & 4 & 0 & 7 & 4 & 2 & 0 & 0 \\
1 & 2 & 3 & 4 & 5 & 7 & 0 & 0 \\
 & & 4 & 9 & 3 & 8 & 2 & 8 \\
 & & 1 & 7 & 2 & 8 & 4 & 0 \\
\hline
8 & 7 & 0 & 8 & \underline{|6} & \underline{5} & \underline{6} & 8
\end{array}
$$

In the third and fourth cases are considered factors which appear as integers, but are in reality decimals; for instance, the sine of 54° is given in the tables as 80902 when in reality it is .80902.

Of interest as regards the use of the word "parabola" is the following (*Clavis*, 1694, p. 19 and the *Clavis* of 1631, p. 8): "The Number found by Division is called the *Quotient*, or also *Parabola*, because it arises out of the Application of a plain Number to a given Longitude, that a congruous Latitude may be found." This is in harmony with etymological dictionaries which speak of a parabola as the application of a given area to a given straight line. The dividend or product is the area; the divisor or factor is the line.

Oughtred gives two processes of long division. The first is identical with the modern process, except that the divisor is written below every remainder, each digit of the divisor being crossed out as soon as it has been used in the partial multiplication. The second method of long division is one of the several types of the old "scratch method." This antiquated process held its place by the side of the modern method in all editions of the *Clavis*. The author divides 467023 by $357\underline{|0926425}$, giving the following instructions: "Take as many of the first Figures of the Divisor as are necessary, for the first Divisor, and then in every following particular Division drop one of the Figures of the Divisor towards the Left Hand, till you have got a competent Quotient." He does not explain abbreviated division as thoroughly as abbreviated multiplication.

$$
\begin{array}{rcl}
 & 17 & \\
 & \cancel{303} & \\
 & \cancel{2803} & \\
 & \cancel{109930} & \\
\underset{\cdots\ \ \cdots}{357\ \underline{|0926}425)} & \cancel{467023} & (1307\ \underline{|80} \\
 & \cancel{357093} & \\
 & \cancel{107127} & \\
 & \cancel{2500} & \\
 & \cancel{286} &
\end{array}
$$

Oughtred does not examine the degree of reliability or accuracy of his processes of abbreviated multiplication and division. Here as in other places he gives in cendensed statement the mode of procedure, without further discussion.

He does not attempt to establish the rules for the addition, subtraction, multiplication and division of positive and negative numbers. "If the Signs are both alike, the Product will be affirmative, if unlike, negative"; then he proceeds to applications. This attitude is superior to that of many writers of the eighteenth and nineteenth centuries, on pedagogical as well as logical grounds: Pedagogically, because the beginner in the study of algebra is not in a position to appreciate an abstract train of thought, as every teacher well knows, and derives better intellectual exercise from the applications of the rules to problems; logically, because the rule of signs in multiplication does not admit of rigorous proof, unless some other assumption is first made which is no less arbitrary than the rule itself. It is well known that the proofs of the rule of signs given by eighteenth-century writers are invalid. Somewhere they involve some surreptitious assumption. This criticism applies even to the proof given by Laplace, which tacitly assumes the distributive law in multiplication.

A word should be said on Oughtred's definition of + and –. He recognizes their double function in algebra by saying (*Clavis,* 1631, p. 2): "*Signum additionis, sive affirmationis, est* + *plus*" and "*Signum subductionis, sive negationis est* – *minus.*" They are symbols which indicate the *quality* of numbers in some instances and *operations* of addition or subtraction in other instances. In the 1694 edition of the *Clavis,* thirty-four years after the death of Oughtred, these symbols are defined as signifying operations only, but are actually used to signify the quality of numbers as well. In this respect the 1694 edition marks a recrudescence.

The characteristic in the *Clavis* that is most striking to a modern reader is the total absence of indices or exponents. There is much discussion in the leading treatises of the latter part of the sixteenth and the early part of the seventeenth centuries on the theory of indices, but modern exponential notation, a^n, is of later date. The modern notation, for positive integral exponents, first appears in Descartes's *Géométrie*, 1637; fractional and negative exponents were first used in the modern form by Sir Isaac Newton, in his announcement of the binomial formula, in a letter written in 1676. This total absence of our modern exponential notation in Oughtred's *Clavis* gives it a strange aspect. Like Vieta, Oughtred uses ordinarily the capital letters, A, B, C,... to designate given numbers; A^2 is written Aq, A^3 is written Ac; for A^4, A^5, A^6 he has, respectively, Aqq, Aqc, Acc. Only on rare occasions, usually when some parallelism in notation is aimed at, does he use small letters[5] to represent numbers or magnitudes. Powers of binomials or polynomials are marked by prefixing the capital letters Q (for square), C (for cube), QQ (for the fourth power), QC (for the fifth power), etc.

Oughtred does not express aggregation by ( ). Parentheses had been used by Girard, and by Clavius as early as 1609,[6] but did not come into general use in mathematical language until the time of Leibniz and by the Bernoullis. Oughtred indicates aggregation by writing a colon (:) at both ends. Thus, Q:A−E: means with his $(A-E)^2$. Similarly, $\sqrt{}$q:A+E: means $\sqrt{(A+E)}$. The two dots at the end are frequently omitted when the part affected includes all the terms of the polynomial to the end. Thus, C:A+B −E=.. means $(a+B-E)^3=..$ There are still further departures from this notation, but they occur so seldom that we incline to the interpretation that they are simply printer's errors. For proportion Oughtred uses the symbol (: :). The proportion $a:b=c:d$ appears in his notation $a\cdot b::c\cdot d$. Apparently, a proportion was not fully recognized in his day as being the expression of an equality of ratios. That probably explains why he did not use = here as in the notation of ordinary equations. Yet Oughtred must have been very close to the interpretation of a proportion as an equality; for he says in his *Elementi decimi Euclidis declaratio, "proportio, sive ratio aequalis* : :" That he introduced this extra symbol, when

[5] See, for instance, Oughtred's *Elementi decimi Euclidis declaratio*, 1652, p. 1, where he uses A and E, and also *a* and *e*.

[6] See *Christophori Clavii Bambergensis Operum mathematicorum, tomus secundus*, Moguntiae, M.DC.XI, Algebra, p. 39.

the one for equality was sufficient, is a misfortune. Simplicity demands that no unnecessary symbols be introduced. However, Oughtred's symbolism is certainly superior to those which preceded. Consider the notation of Clavius.[7] He wrote $20:60=4:x$, $x=12$, thus: "20.60.4? *fiunt* 12." The insufficiency of such a notation in the more involved expressions frequently arising in algebra is readily seen. Hence Oughtred's notation (: :) was early adopted by English mathematicians. It was used by John Wallis at Oxford, by Samuel Foster at Gresham College, by James Gregory of Edinburgh, by the translators into English of Rahn's algebra and by many other early writers. Oughtred has been credited generally with the introduction of St. Andrews's cross × as the symbol for multiplication in the *Clavis* of 1631. We have discovered that this symbol, or rather the letter *x* which closely resembles it, occurs as the sign of multiplication thirteen years earlier in an anonymous "Appendix to the Logarithmes, shewing the practise of the Calculation of Triangles etc." to Edward Wright's translation of John Napier's *Descriptio*, published in 1618. Later we shall give our reasons for believing that Oughtred is the author of that "Appendix." The × has survived as a symbol of multiplication.

Another symbol introduced by Oughtred and found in modern book is $\sim$, expressing difference; thus $C \sim D$ signifies the difference between C and D, even when D is the larger number.[8] This symbol was used by John Wallis in 1657.[9]

Oughtred represented in symbols also certain composite expressions, as for instance $A+E=Z$, $A-E=X$, where A is greater than E. He represented by a symbol also each of the following: A^2+E^2, A^3+E^3, A^2-E^2, A^3-E^3.

Oughtred practically translated the 10th book of Euclid from its ponderous rhetorical form into that of brief symbolism. An appeal to the eye was a passion with Oughtred. The present writer has collected the different mathematical symbols used by Oughtred and has found more than one hundred and fifty of them.

The differences between the seven different editions of the *Clavis* lie mainly in the special parts appended to some editions and dropped in the latest editions. The part which originally con-

[7] *Christophori Clavii operum mathematicorum Tomus Secundus*, Moguntiae, M.DC.XI., *Epitome arithmeticae*, p. 36.

[8] See *Elementi decimi Euclidis declaratio*, 1652, p. 2.

[9] See Johannis Wallisii *Operum mathematicorum pars prima*, Oxonii, 1657, p. 247.

stituted the *Clavis* was not materially altered, except in two or three of the original twenty chapters. These changes were made in the editions of 1647 and 1648. After the first edition, great stress was laid upon the theory of indices upon the very first page as also in passages further on. Of course, Oughtred did not have our modern notation of indices or exponents, but their theory had been a part of algebra and arithmetic for some time. Oughtred incorporated this theory in his brief exposition of the Hindu-Arabic notation and in his explanation of logarithms. As previously pointed out, the last three chapters of the 1631 edition were considerably rearranged in the later editions and combined into two chapters, so that the *Clavis* proper had nineteen chapters instead of twenty in the editions after the first. These chapters consisted of applications of algebra to geometry and were so framed as to constitute a severe test of the student's grip of the subject. The very last problem deals with the division of angles into equal parts. He derives the cubic equation upon which the trisection depends algebraically, also the equations of the fifth degree and seventh degree upon which the divisions of the angle into 5 and 7 equal parts depend, respectively. The exposition was severely brief, yet accurate. He did not believe in conducting the reader along level paths or along slight inclines. He was a guide for mountain climbers and woe unto him who lacked nerve.

Oughtred lays great stress upon expansions of powers of a binomial. He makes use of these expansions in the solution of numerical equations. To one who does not specialize in the history of mathematics such expansions may create surprise, for did not Newton invent the binomial theorem after the death of Oughtred? As a matter of fact, the expansions of positive integral powers of a binomial were known long before Newton, not only to seventeenth-century but even sixteenth-century mathematicians. Oughtred's *Clavis* of 1631 gave the binomial coefficients for all powers up to and including the tenth. What Newton really accomplished was the generalization of the binomial expansion which makes it applicable to negative and fractional exponents and converts it into an infinite series.

As a specimen of Oughtred's style of writing we quote his solution of quadratic equations, accompanied by a translation into English and into modern mathematical symbols.

As a preliminary step he lets[10]

[10] *Clavis* of 1631, Chap. XIX, sect. 5, p. 50.

$$Z = A + E \quad \text{and} \quad A > E;$$

he lets also $X = A - E$. From these relations he obtains identities which, in modern notation, are $\frac{1}{4}Z^2 - AE = \frac{1}{4}(\frac{1}{2}Z - E)^2 = \frac{1}{4}X^2$. Now, if we know Z and AE, we can find $\frac{1}{2}X$. Then $\frac{1}{2}(Z+X) = A$, and $\frac{1}{2}(Z-X) = E$, and

$$A = \tfrac{1}{2}Z + \sqrt{\tfrac{1}{4}Z^2 - AE.}$$

Having established these preliminaries, he then proceeds. (We translate the Latin passage, using the modern exponential notation and parentheses.)

"Given therefore an unequally divided line Z (10), and a rectangle beneath the segments AE (21) which is a gnomon. Half the difference of the segments $\frac{1}{2}X$ is given, and consequently the segment itself. For, if one of the two segments is placed equal to A, the other will be $Z - A$. Moreover, the rectangle is $ZA - A^2 = AE$. And because Z and AE are given, and there is $\frac{1}{4}Z^2 - AE = \frac{1}{4}X^2$, and by 5c. 18, $\frac{1}{2}Z + \frac{1}{2}X = A$, and $\frac{1}{2}Z - \frac{1}{2}X = E$, the equation will be solved thus: $\frac{1}{2}Z \pm \sqrt{(\frac{1}{4}Z^2 - AE)} = A \begin{cases}\text{major segment}\\ \text{minor segment.}\end{cases}$

"And so an equation having been proposed in which three species (terms) are in equally ascending powers, the highest species, moreover, being negative, the given magnitude which constitutes the middle species is the line to be bisected. And the given absolute magnitude to which it is equal is the rectangle beneath the unequal segments, without gnomon. As $ZA - A^2 = AE$, or in numbers, $10x - x^2 = 21$. And A or x is one of the two unequal segments. It may be found thus:

"The half of the middle species is $Z^2/2$ (5). its square is $Z^2/4$ (25). From it subtract the absolute term AE (21), and $(Z^2/4) - A^2$ (4) will be the square of half the difference of the segments. The square root of this, $\sqrt{[(Z^2/2)^2 - AE]}$ (2) is half the difference. If you add it to half the coefficient $Z/2$ (5), the longer segment is obtained, if you subtract it, the smaller segment is obtained. I say:

$$(Z/2) + [\sqrt{(Z^2/4 - AE)}] = A \begin{cases}\text{major segment}\\ \text{minor segment."}\end{cases}$$

The quadratic equation $Aq + ZA = AE$ receives similar treatment. This and the preceding equation, $ZA - Aq = AE$, constitute together a solution of the general quadratic equation, $x^2 + ax = b$, provided that E or Z are not restricted to positive values, but admit

of being either positive or negative, a case not adequately treated by Oughtred. Imaginary numbers and imaginary roots receive no consideration whatever.

A notation suggested by Vieta and favored by Girard made vowels stand for unknowns and consonants for knowns. This conventionality was adopted by Oughtred in parts of his algebra, but not throughout. Near the beginning he used Q to designate the unknown, though usually this letter stood with him for the "square" of the expression after it.[11]

We quote the description of the *Clavis* that was given by Oughtred's greatest pupil, John Wallis. It contains additional information of interest to us. Wallis devotes Chapter XV of his *Treatise of Algebra*, London, 1685, pp. 67-69, to Mr. Oughtred and his *Clavis*, saying:

"Mr. William Oughtred (our Country-man) in his *Clavis Mathematicae*, (or Key of Mathematicks,) first published in the Year 1631, follows Vieta (as he did Diophantus) in the use of the Cossick Denominations; omitting (as he had done) the names of *Sursolids*, and contenting himself with those of *Square* and *Cube*, and the Compounds of these.

"But he doth abridge Vieta's Characters or Species, using only the letters q, c, &c. which in Vieta are expressed (at length) by *Quadrate, Cube*, &c. For though when Vieta first introduced this way of Specious Arithmetick, it was more necessary (the thing being new,) to express it in words at length: Yet when the thing was once received in practise, Mr. Oughtred (who affected brevity, and to deliver what he taught as briefly as might be, and reduce all to a short view,) contented himself with single Letters instead of Those words.

"Thus what Vieta would have written

$$\frac{\text{A } \textit{Quadrate}\text{, into B } \textit{Cube,}}{\text{C D E } \textit{Solid,}} \ \textit{Equal to} \text{ F. G. } \textit{Plane,}$$

would with him be thus expressed

$$\frac{A_q \, B_c}{C\,D\,E} = FG.$$

"And the better to distinguish upon the first view, what quantities were Known, and what Unknown, he doth (usually) denote the Known by *Consonants*, and the Unknown by *Vowels*; as Vieta (for the same reason) had done before him.

[11] We have noticed the representation of known quantities by consonants and the unknown by vowels in Wingate's *Arithmetick made easie*, edited by

"He doth also (to very great advantage) make use of several Ligatures, or Compendious Notes, to signify *Summs, Differences,* and *Rectangles* of several Quantities. As for instance, Of two Quantities A (the Greater), and E (the Lesser), the Sum he calls Z, the Difference X, the Rectangle AE...."

"Which being of (almost) a constant signification with him throughout, do save a great circumlocution of words, (each Letter serving instead of a Definition;) and are also made use of (with very great advantage) to discover the true nature of divers intricate Operations, arising from the various compositions of such Parts, Sums, Differences, and Rectangles; (of which there is great plenty in his *Clavis,* Cap. 11, 16, 18, 19. and elsewhere,) which without such Ligatures, or CompendiousNotes, would not be easily discovered or apprehended..

"In know there are who find fault with his *Clavis,* as too obscure, because so short, but without cause; for his words be always full, but not Redundant, and need only a little attention in the Reader to weight the force of every word, and the Syntax of it;... And this, when once apprehended, is much more easily retained, than if it were expressed with the prolixity of some other Writers; where a Reader must first be at the pains to weed out a great deal of superfluous Language, that he may have a short prospect of what is material; which is here contracted for him in short Synopsis"...

"Mr. Oughtred in his *Clavis,* contents himself (for the most part) with the solution of Quadratick Equations, without proceeding (or very sparingly) to Cubick Equations, and those of Higher Powers; having designed that Work for an *Introduction* into *Algebra* so far, leaving the Discussion of Superior Equations for another work.... He contents himself likewise in Resolving Equations, to take notice of the *Affirmative* or *Positive Roots*; omitting the *Negative* or *Ablative* Roots, and such as are called *Imaginary* or *Impossible Roots.* And of those which he calls *Ambiguous* Equations, (as having more Affirmative Roots than one,) he doth not (that I remember) any where take notice of more than *Two* Affirmative Roots: (Because in Quadratick Equations, which are those he handleth, there are indeed no more.) Whereas yet in *Cubick* Equations, there may be *Three,* and in those of Higher Powers, yet more. Which Vieta was well aware of, and men-

John Kersey, London, 1650, algebra, p. 382; and in the second part, section 19, of Jonas Moore's *Arithmetick in two parts,* London, 1660, second part; Moore suggests as an alternative the use of *z, y, x,* etc. for the unknowns. The practice of representing unknowns by vowels did not spread widely in England.

tioneth in some of his Writings; and of which Mr. Oughtred could not be ignorant."

OUGHTRED'S CIRCLES OF PROPORTION AND TRIGONOMETRY.

Oughtred wrote and had published three important mathematical books, the *Clavis,* the *Circles of Proportion*[12] and a *Trigonometry.*[13] This last appeared in the year 1657 at London, in both Latin and English.

It is claimed that the trigonometry was "neither finished nor published by himself, but collected out of his scattered papers; and though he connived at the printing it, yet imperfectly done, as appears by his MSS.; and one of the printed Books, corrected by his own Hand."[14] Doubtless more accurate on this point is a letter of Richard Stokes who saw the book through the press:[15]

"I have procured your Trigonometry to be written over in a fair hand, which when finished I will send to you, to know if it be according to your mind; for I intend (since you were pleased to give your assent) to endeavour to print it with Mr. Briggs his Tables, and so soon as I can get the Prutenic Tables I will turn those of the sun and moon, and send them to you."

In the preface to the Latin edition Stokes writes:

"Since this trigonometry was written for private use without the intention of having it published, it pleased the Reverend Author, before allowing it to go to press, to expunge some things, to change

[12] There are two title-pages to the edition of 1632. The first title-page is as follows: *The Circles of Proportion and The Horizontall Instrument. Both invented, and the uses of both Written in Latine by Mr. W. O. Translated into English: and set forth for the publique benefit by William Forster. London. Printed for Elias Allen maker of these and all other mathematical Instruments, and are to be sold at his shop over against St. Clements church with out Temple-barr. 1632. T. Cecill Sculp.*

In 1633 there was added the following, with a separate title-page: *An addition unto the Use of the Instrument called the Circles of Proportion. London, 1633,* this being followed by Oughtred's *To the English Gentrie etc.* In the British Museum there is a copy of another impression, dated 1639, with the *Addition unto the use of the Instrument etc.,* bearing the original date, 1633, and with the epistle, *To the English Gentrie* etc., inserted immediately after Forster's dedication, instead of at the end of the volume.

[13] The complete title of the English edition is as follows: *Trigonometrie, or, The manner of calculating the Sides and Angles of Triangles, by the Mathematical Canon, demonstrated. By William Oughtred Etoneus. And published by Richard Stokes Fellow of Kings Colledge in Cambridge, and Arthur Haughton Gentleman. London, Printed by R. and W. Leybourn, for Thomas Johnson at the Golden Key in St. Paul's Church-yard.* M.DC.LXII.

[14] Jer. Collier, *The Great Historical, Geographical, Genealogical and Poetical Dictionary,* Vol. II, London, 1701, art. "Oughtred."

[15] Rigaud, *op. cit.,* Vol. I, p. 82.

other things and even to make some additions and insert more lucid methods of exposition."

This much is certain, the *Trigonometry* bears the impress characteristic of Oughtred. Like all his mathematical writings, the book was very condensed. Aside from the tables, the text covered only 36 pages. Plane and spherical triangles were taken up together. The treatise is known in the history of trigonometry as among the very earliest works to adopt a condensed symbolism so that equations involving trigonometric functions could be easily taken in by the eye. In the work of 1657 contractions are given as follows. *s* = sine, *t* = tangent, *se* = secant, *s co* = cosine (sine complement), *t co* = cotangent, *se co* = cosecant, *log* = logarithm, *Z cru* = sum of the sides of a rectangle or right angle, *X cru* = difference of these sides. It has been generally overlooked by historians that Oughtred used the abbreviations of trigonometric functions named above, a quarter of a century earlier, in his *Circles of Proportion,* 1632, 1633. Moreover, he used sometimes also the abbreviations which are current at the present time, namely sin = sine, tan = tangent, sec = secant. We know that the *Circles of Proportion* existed in manuscript many years before they were published. The symbol *sv* for *sinus versus* occurs in the *Clavis* of 1631. The great importance of well-chosen symbols needs no emphasis to readers of the present day. With reference to Oughtred's trigonometric symbols, Augustus De Morgan said:[16] "This is so very important a step, simple as it is, that Euler is justly held to have greatly advanced trigonometry by its introduction. Nobody that we know of has noticed that Oughtred was master of the improvement, and willing to have taught it, if people would have learnt." We find, however, that even Oughtred cannot be given the whole credit in this matter. As early as 1624, the contractions *sin* for sine and *tan* for tangent appear on the drawing representing Gunter's scale, but Gunter did not use them in his books, except in the drawing of his scale.[17] A closer competitor for the honor of first using these trigonometric abbreviations is Richard Norwood in his *Trigonometrie,* London, 1631, where *s* stands for sine, *t* for tangent, *sc* for sine complement (cosine), *tc* for tangent complement (cotangent), and *sec* for secant. Norwood was a teacher of mathematics in London and a well-known

[16] A. De Morgan, *Budget of Paradoxes,* London, 1872, p. 451; 2d edition, Chicago, 1915, Vol. II, p. 303.

[17] E. Gunter, *Description and Use of the Sector, the Crosse-staffe and other Instruments,* London, 1624. The second book, p. 31.

writer of books on navigation. Aside from the abbreviations just cited, Norwood did not use nearly as much symbolism in his mathematics as did Oughtred. The innovation of designating the sides and angles of a triangle by A, B, C and *a, b, c,* so that A was opposite *a,* B opposite *b,* and C opposite *c,* is attributed to Leonard Euler (1753), but was first used by Richard Rawlinson of Queen's College, Oxford, sometime after 1655 and before 1668. Oughtred did not use Rawlinson's notation.[18]

Mention should be made of trigonometric symbols used even earlier than any of the preceding, in "An Appendix to the Logarithmes, shewing the practise of the Calculation of Triangles, etc." printed in Edward Wright's edition of Napier's *A Description of the Admirable Table of Logarithmes,* London, 1618. We referred to this "Appendix" in tracing the origin of the sign ×. It contains, on page 4, the following passage: "For the Logarithme of an arch or an angle I set before (s), for the antilogarithme or compliment thereof (s*) and for the Differential (t)." In further explanation of this rather unsatisfactory passage, the author (Oughtred?) says, "As for example: sB+BC=CA. that is, the Logarithme of an angle B. at the Base of a plane right-angled triangle, increased by the addition of the Logarithm of BC, the hypothenuse thereof, is equall to the Logarithme of CA the cathetus."

Here "logarithme of an angle B" evidently means "log sin B," just as with Napier, "Logarithms of the arcs" signifies really "Logarithms of the sines of the angles." In Napier's table, the numbers in the column marked "Differentiae" signify log. sine minus log. cosine of an angle; that is, the logarithms of the tangents. This explains the contraction (t) in the "Appendix." The conclusion of all this is that as early as 1618 the signs *s, s**, *t* were used for ***sine, cosine,*** and ***tangent,*** respectively.

In trigonometry English writers of the first half of the seventeenth century used contractions more freely than their continental contemporaries, yea even more freely than English writers of a later period. Von Braunmühl, the great historian of trigonometry, gives Oughtred much praise for his trigonometry, and points out that half a century later the army of writers on trigonometry had hardly yet reached the standard set by Oughtred's[19] analysis. Oughtred must be credited also with the first complete proof that

[18] F. Cajori, "On the History of a Notation in Trigonometry" in *Nature,* Vol. 94, 1915, pp. 642, 643.

[19] A. v. Braunmühl, *Geschichte der Trigonometrie,* 2. Teil, Leipsic, 1903, pp. 42, 91.

was given to the first two of "Napier's analogies." His trigonometry contains seven-place tables of sines, tangents and secants, and six-place tables of logarithmic sines and tangents; also seven-place logarithmic tables of numbers. At the time of Oughtred there was some agitation in favor of a wider introduction of decimal systems. This movement is reflected in these tables which contain the centesimal division of the degree, a practice which is urged for general adoption in our own day, particularly by the French.

SOLUTION OF NUMERICAL EQUATIONS.

In the solution of numerical equations Oughtred does not mention the sources from which he drew, but the method is substantially that of the great French algebraist Vieta, as explained in a publication which appeared in 1600 in Paris under the title, *De numerosa potestatum purarum atque adfectarum ad exegesin resolutione tractatus.* In view of the fact that Vieta's process has been described inaccurately by leading modern historians including H. Hankel[20] and M. Cantor,[21] it may be worth while to go into some detail.[22] By them it is made to appear as identical with the procedure given later by Newton. The two are not the same. The difference lies in the divisor used. What is now called "Newton's method" is Newton's method as modified by Joseph Raphson.[23] The Newton-Raphson method of approximation to the roots of an equation $f(x)=0$ is usually given the form $a-[f(a)/f'(a)]$, where a is an approximate value of the required root. It will be seen that the divisor is $f'(a)$. Vieta's divisor is different; it is

$$|f(a+s_1)-f(a)|-s^n,$$

where $f(x)$ is the left of the equation $f(x)=k$, n is the degree of equation and s_1 is a unit of the denomination of the digit next to be found. Thus in $x^3+420000x=247651713$, it can be shown that 417 is approximately a root; suppose that a has been taken to be 400, then $s_1=10$; but if, at the next step of approximation, a is taken

[20] H. Hankel, *Geschichte der Mathematik im Alterthum und Mittelalter,* Leipsic, 1874, pp. 369, 370.

[21] M. Cantor, *Vorlesungen über die Geschichte der Mathematik,* II, 1900, pp. 640, 641.

[22] This matter has been discussed in a paper "A History of the Arithmetical Methods of Approximation etc." by F. Cajori, in the *Colorado College Publication,* General Series No. 51, 1910, pp. 182-184. Later this subject was again treated by G. Eneström in *Bibliotheca mathematica,* 3. Folge, Vol. 11, 1911, pp. 234, 235.

[23] See F. Cajori, *loc. cit.,* p. 193.

to be 410, then $s_1 = 1$. In this example, taking $a = 400$, Vieta's divisor would have been 9120000; Newton's divisor would have been 900000.

A comparison of Vieta's method with the Newton-Raphson method reveals the fact that Vieta's divisor is more reliable, but labors under the very great disadvantage of requiring a much larger amount of computation. The latter divisor is accurate enough and easier to compute. Altogether the Newton-Raphson process marks a decided advance over that of Vieta.

As already stated, it is the method of Vieta that Oughtred explains. The Englishman's exposition is an improvement on that of Vieta, printed forty years earlier. Nevertheless, Oughtred's explanation is far from easy to follow. The theory of equations was at that time still in its primitive stage of development. Algebraic notation was not sufficiently developed to enable the argument to be condensed into a form easily surveyed. So complicated does Vieta's process of approximation appear, that M. Cantor failed to recognize that Vieta possessed a uniform mode of procedure. But when one has in mind the general expression for Vieta's divisor which we gave above, one will recognize that there was marked uniformity in Vieta's approximations.

Oughtred allows himself twenty-eight sections in which to explain the process and at the close cannot forbear remarking that 28 is a "perfect" number (being equal to the sum of its divisors, 1, 2, 4, 7, 14).

The early part of his exposition shows how an equation may be transformed so as to make its roots 10, 100, 1000 or 10^m times smaller. This simplifies the task of "locating a root"; that is, of finding between what integers the root lies.

Taking one of Oughtred's equations, $x^4 - 72x^3 + 238600x = 8725815$, upon dividing $72x^3$ by 10, $238600x$ by 1000, and 8725815 by 10,000, we obtain $x^4 - 7 \cdot 2x^3 + 238 \cdot 6x = 872 \cdot 5$. Dividing both sides by x, we obtain $x^3 + 238 \cdot 6 - 7 \cdot 2x^2 = x)872 \cdot 5$. Letting $x = 4$, we have $64 + 238 \cdot 6 - 115 \cdot 2 = 187 \cdot 4$.

But $4)872 \cdot 5(218 \cdot 1$; 4 is too small. Next let $x = 5$, we have $125 + 238 \cdot 6 - 180 = 183 \cdot 6$.

But $5)872 \cdot 5(174 \cdot 5$; 5 is too large. We take the lesser value, $x = 4$, or in the original equation, $x = 40$. This method may be used to find the second digit in the root. Oughtred divides both sides of the equation by x^2, and obtains $x^2 + x)238600 - 72x = x^2)8725815$. He tries $x = 47$ and $x = 48$, and finds that $x = 47$.

He explains also how the last computation may be done by logarithms. Thereby he established for himself the record of being the first to use logarithms in the solution of affected equations.

"Exemplus II.

$$1 + 420000_l = 247651713$$

Hoc est, $L_c + C_q L = D_c$.

	247	651	713	(417
	42	000	0	C_q
	64			A_c
	168	000	0	C_q A
	232	000	0	Ablatit.
R	15	651	713	
	4	8		3 A_q
		12		3 A
	4	200	00	C_q
	9	120	00	Divisor.
	4	8		3 A_q E
		12		3 A E_q
		1		E_c
	4	200	00	C_q E
	9	121	00	Ablatit.
R	6	530	713	
		504	3	3 A_q
		1	23	3 A
		420	000	C_q
		925	530	Divisor.
	3	530	1	3 A_q E
		60	27	3 A E_q
			343	E_c
	2	940	000	C_q E
	6	530	713	Ablatit."

4	1
16	8
	1
16	8 1

As an illustration of Oughtred's method of approximation, after the root sought has been located, we choose for brevity a cubic in preference to a quartic. We select the equation $x^3 +$

$420000x = 247651713$. By the process explained above a root is found to lie between $x = 400$ and $x = 500$. From this point on, the approximation as given by Oughtred is as shown on previous page.

In further explanation of this process, observe that the given equation is of the form $L_c = C_qL = D_c$, where L is our x, $C_q = 420000$, $D_c = 247651713$. In the first step of approximation, let $L = A + E$, where $A = 400$ and E is, as yet, undetermined. We have $L_c = (A+E)^3 = A^3 + 3A^2E + 3AE^2 + E^3$
and $C_qL = 420000\,(A+E)$.
Subtract from 247651713 the sum of the known terms A^3 (his A_c) and 420000 A (his C_qA). This sum is 232000000; the remainder is 15651713.

Next, he evaluates the coefficients of E in $3\,A^2E$ and 420000 E, also 3A, the coefficient of E^2. He obtains $3A^2 = 480000$, $3A = 1200$, $C_q = 420000$. He interprets $3A^2$ and C_q as tens, 3A as hundreds. Accordingly, he obtains as their sum 9120000, which is the *Divisor* for finding the second digit in the approximation. Observe that this divisor is the value of $|f(a+s_1) - f(a)| - s_1^n$ in our general expression, where $a = 400$, $s_1 = 10$, $n = 3$, $f(x) = x^3 + 420000x$.

Dividing the remainder 15651713 by 9120000, he obtains the integer 1 in tens place; thus $E = 10$, approximately. He now computes the terms $3A^2E$, $3AE^2$ and E^3 to be respectively, 4800000, 120000, 1000. Their sum is 9121000. Subtracting it from the previous remainder 15651713, leaves the new remainder, 6530713.

From here on each step is a repetition of the preceding step. The new A is 410, the new E is to be determined. We have now in closer approximation, $L = A + E$. This time we do not subtract A^3 and C_qA, because this subtraction is already affected by the preceding work.

We find the second trial divisor by computing the sum of $3A^2$, 3A and C_q; that is, the sum of 504300, 1230, 420000, which is 925530. Again, this divisor can be computed by our general expression for divisors, by taking $a = 410$, $s_1 = 1$, $n = 3$.

Dividing 6530713 by 925530 yields the integer 7. Thus $E = 7$. Computing $3A^2E$, $3AE^2$, E^2 and subtracting their sum, the remainder is 0. Hence 417 is an exact root of the given equation.

Since the extraction of a cube root is merely the solution of a pure cubic equation, $x^3 = n$, the process given above may be utilized in finding cube roots. This is precisely what Oughtred does in Chapter XIV of his *Clavis*. If the above computation is modified by

putting $C_q = 0$, the process will yield the approximate cube root of 247651713.

Oughtred solves 16 examples by the process of approximation here explained. Of these, 9 are cubics, 5 are quartics, and 2 are quintics. In all cases he finds only one or two real roots. Of the roots sought, five are irrational, the remaining are rational and are computed to their exact values. Three of the computed roots have 2 figures each, 9 roots have 3 figures each, 4 roots have 4 figures each. While no attempt is made to secure all the roots—methods of computing complex roots were invented much later—he computes roots of equations which involve large coefficients and some of them are of a degree as high as the fifth. In view of the fact that many editions of the *Clavis* were issued, one impression as late as 1702, it contributed probably more than any other book to the popularization of Vieta's method in England.

Before Oughtred, Thomas Harriot and William Milbourn are the only Englishmen known to have solved numerical equations of higher degrees. Milbourn published nothing. Harriot slightly modified Vieta's process by simplifying somewhat the formation of the trial divisor. This method of approximation was the best in existence until the publication by Wallis in 1685 of Newton's method of approximation.

LOGARITHMS.

Oughtred's treatment of logarithms is quite in accordance with the more recent practice.[24] He explains the finding of the *index* (our *characteristic*); he states that "the sum of two Logarithms is the Logarithm of the Product of their Valors; and their difference is the Logarithm of the Quotient," that "the Logarithm of the side [436] drawn upon the Index number [2] of dimensions of any Potestas is the logarithm of the same Potestas" [436^2], that "the logarithm of any Potestas [436^2] divided by the number of its dimensions [2] affordeth the Logarithm of its Root [436]." These statements of Oughtred occur for the first time in the *Key of the Mathematicks* of 1647; the *Clavis* of 1631 contains no treatment of logarithms.

If the characteristic of a logarithm is negative, Oughtred indicates this fact by placing the = *above* the characteristic. He separates the characteristic and mantissa by a *comma*, but still uses

[24] See William Oughtred's *Key of the Mathematicks*, London, 1494, pp. 173-175, tract, "Of the Resolution of the Affected Equations," or any edition of the *Clavis* after the first.

the sign ⌊ to indicate decimal fractions. He uses the contraction "log."

INVENTION OF THE SLIDE RULE; CONTROVERSY ON PRIORITY OF INVENTION.

Oughtred's most original line of scientific activity is the one least known to the present generation. Augustus De Morgan, in speaking of Oughtred who was sometimes called "Oughtred Aetonensis," remarks: "He is an animal of extinct race, an Eton mathematician. Few Eton men, even of the minority which knows what a sliding rule is, are aware that the inventor was of their own school and college."[25] The invention of the slide rule has, until recently,[26] been a matter of dispute; it has been erroneously ascribed to Edmund Gunter, Edmund Wingate, Seth Partridge and others. We have been able to establish that William Oughtred was the first inventor of slide rules, though not the first to publish thereon. We shall see that Oughtred invented slide rules about 1622, but the descriptions of his instruments were not put into print before 1632 and 1633. Meanwhile one of his own pupils, Richard Delamain, who probably invented the circular slide rule independently, published a description in 1630, at London, in a pamphlet of 32 pages entitled *Grammelogia; or the Mathematicall Ring.* In editions of this pamphlet which appeared during the following three or four years, various parts were added on, and some parts of the first and second editions eliminated. Thus Delamain antedates Oughtred two years in the publication of a description of a circular slide rule. But Oughtred had invented also a rectilinear slide rule, a description of which appeared in 1633. To the invention of this Oughtred has a clear title. A bitter controversy sprang up between Delamain on one hand, and Oughtred and some of his pupils on the other, on the priority and independence of invention of the circular slide rule. Few inventors and scientific men are so fortunate as to escape contests. The reader needs only to recall the disputes which have arisen, involving the researches of Sir Isaac Newton and Leibniz on the differential and integral calculus, of Thomas Harriot and René Descartes relating to the theory of equations, of Robert Mayer, Hermann v. Helmholtz and Joule on the principle of the conservation of energy, or of Robert Morse, Joseph Henry, Gauss and Weber, and others on the telegraph, to see that questions of

[25] A. De Morgan, *op. cit.*, p. 451; 2d ed., II, p. 303.

[26] See F. Cajori, *History of the Logarithmic Slide Rule*, New York, 1909, pp. 7-14, Addenda, p. ii.

priority and independence are not uncommon. The controversy between Oughtred and Delamain embittered Oughtred's life for many years. He refers to it in print on more than one occasion. We are preparing a separate article giving the details of this controversy and shall confine ourselves at present to the statement that it is by no means clear that Delamain stole the invention from Oughtred; Delamain was probably an independent inventor. Moreover, it is highly probable that the controversy would never have arisen, had not some of Oughtred's pupils urged and forced him into it. William Forster stated in the preface to the *Circles of Proportion* of 1632 that while he had been carefully preparing the manuscript for the press, "another to whom the Author [Oughtred] in a louing confidence discovered this intent, using more hast then good speed, went about to preocupate." It was this passage which started the conflagration. Another pupil, W. Robinson, wrote to Oughtred, when the latter was preparing his *Apologeticall Epistle* as a reply to Delamain's counter-charges:[27] "Good sir, let me be beholden to you for your Apology whensoever it comes forth, and (if I speak not too late) let me entreat you, whip ignorance well on the blind side, and we may turn him round, and see what part of him is free." As stated previously, Oughtred's circular slide rule was described by him in his *Circles of Proportion*, London, 1632, which was translated from Oughtred's Latin manuscript and then seen through the press by his pupil, William Forster. In 1633 appeared *An Addition unto the Use of the Instrument called the Circles of Proportion* which contained at the end "The Declaration of the two Rulers for Calculation," giving a description of Oughtred's rectilinear slide rule. This *Addition* was bound with the *Circles of Proportion* as one volume. About the same time Oughtred described a modified form of the rectilinear slide rule, to be used in London for gauging.[28]

MINOR WORKS.

Among the minor works of Oughtred must be ranked his booklet of forty pages to which reference has already been made, entitled, *The New Artificial Gauging Line or Rod,* London, 1633. His different designs of slide rules and his inventions of sun-dials as well as his exposition of the making of watches show that he displayed unusual interest and talent in the various mathematical

[27] Rigaud, *op. cit.,* Vol. I, p. 12.

[28] *The New Artificial Gauging Line or Rod: together with rules concerning the use thereof: Invented and written by William Oughtred.* London, 1633.

instruments. A short tract on watch-making was brought out in London as an appendix to the *Horological Dialogues* of a clock and watch maker who signed himself "J. S." (John Smith?). Oughtred's tract appeared with its own title-page, but with pagination continued from the preceding part, as *An Appendix wherein is contained a Method of Calculating all Numbers for Watches. Written originally by that famous Mathematician Mr. William Oughtred, and now made Publick. By J. S. of London, Clock-maker.* London, 1675.

"J. S." says in his preface:

"The method following was many years since Compiled by Mr. Oughtred for the use of some Ingenious Gentlemen his friends, who for recreation at the University, studied to find out the reason and Knowledge of Watch-work, which seemed also to be a thing with which Mr. Oughtred himself was much affected, as may in part appear by his putting out of his own Son to the same Trade, for whose use (as I am informed) he did compile a larger tract, but what became of it cannot be known."

Notwithstanding Oughtred's marked activity in the design of mathematical instruments, and his use of surveying instruments, he always spoke in depreciating terms of their importance and their educational value. In his epistle against Delamain he says:[29]

"The Instruments I doe not value or weigh one single penny. If I had been ambitious of praise, or had thought them (or better then they) worthy, at which to have taken my rise, out of my secure and quiet obscuritie, to mount up into glory, and the knowledge of men: I could have done it many yeares before...

"Long agoe, when I was a young student of the Mathematicall Sciences, I tryed many wayes and devices to fit my selve with some good Diall or Instrument portable for my pocket, to finde the houre, and try other conclusions by, and accordingly framed for that my purpose both Quadrants, and Rings, and Cylinders, and many other composures. Yet not to my full content and satisfaction; for either they performed but little, or els were patched up with a diversity of lines by an unnaturall and forced contexture. At last I.... found what I had before with much studie and paines in vaine sought for."

Mention has been made on the previous pages of two of his papers on sun-dials, and prepared (as he says) when he was in his twenty-third year, and was first published in the *Clavis* of 1647. The second paper appeared in his *Circles of Proportion.*

[29] W. Oughtred, *Apologeticall Epistle,* p. 13

Both before and after the time of Oughtred much was written on sun-dials. Such instruments were set up against the walls of prominent buildings, much as the faces of clocks in our time. The inscriptions that were put upon sun-dials are often very clever: "I count only the hours of sunshine," "Alas, how fleeting." A sun-dial on the grounds of Merchiston Castle, in Edinburgh, where the inventor of logarithms, John Napier, lived for many years, bears the inscription, "Ere time be tint, tak tent of time" (Ere time be lost, take heed of time).

Portable sun-dials were sometimes carried in pockets, as we carry watches. Thus Shakespeare, in *As You Like It,* Act II, Sc. 7:

"And he drew a diall from his poke."

Watches were first made for carrying in the pocket about 1658.

Because of this literary, scientific and practical interest in methods of indicating time it is not surprising that Oughtred devoted himself to the mastery and the advancement of methods of time-measurement.

Besides the accounts previously noted, there came from his pen: *The Description and Use of the double Horizontall Dyall: Whereby not onely the hower of the day is shewne; but also the Meridian Line is found: And most Astronomical Questions, which may be done by the Globe, are resolved. Invented and written by W. O.,* London, 1636.

The "Horizontall Dyall" and "Horologicall Ring" appeared again as appendices to Oughtred's translation from the French of a book on mathematical recreations.

The fourth French edition of that work appeared in 1627 at Paris, under the title of *Recreations mathematiqve,* written by "Henry van Etten," a pseudonym for the French Jesuit Jean Leurechon (1591-1690). English editions appeared in 1633, 1653 and 1674. The full title of the 1653 edition conveys an idea of the contents of the text:

Mathematicall Recreations, or, A Collection of many Problemes, extracted out of the Ancient and Modern Philosophers, as Secrets and Experiments in Arithmetick, Geometry, Cosmographie, Musick, Opticks, Architecture, Statick, Mechanicks, Chemistry, Waterworks, Fire-works, &c. Not vulgarly manifest till now. Written first in Greek and Latin, lately compil'd in French, by Henry Van Etten, and now in English, with the Examinations and Augmenta-

tions of divers Modern Mathematicians. Whereunto is added the Description and Use of the Generall Horologicall Ring. And The Double Horizontall Diall. Invented and written by William Oughtred. London, Printed for William Leake, at the Signe of the Crown in Fleet-street, between the two Temple-Gates. MDCLIII.

The graphic solution of spherical triangles by the accurate drawing of the triangles on a sphere and the measurement of the unknown parts in the drawing, was explained by Oughtred in a short tract which was published by his son-in-law, Christopher Brookes, under the following title:

The Solution of all Sphaerical Triangles both right and oblique By the Planisphaere: Whereby two of the Sphaerical partes sought, are at one position most easily found out. Published with consent of the Author, By Christopher Brookes, Mathematique Instrument-maker, and Manciple of Wadham Colledge, in Oxford.

Brookes says in the preface: "I have oftentimes seen my Reverend friend Mr. W. O. in his resolution of all sphaericall triangles both right and oblique, to use a planisphaere, without the tedious labour of Trigonometry by the ordinary Canons: which planisphaere he had delineated with his own hands, and used in his calculations more than Forty years before."

Interesting as one of the sources from which Oughtred obtained his knowledge of the conic sections is his study of Mydorge. A tract which he wrote thereon was published by Jonas Moore, in his *Arithmetick in two books*.... [containing also] *the two first books of Mydorgius his conical sections analyzed by that reverend devine Mr. W. Oughtred, Englished and completed with cuts.* London, 1660. Another edition bears the date 1688.

To be noted among the minor works of Oughtred are his posthumous papers. He left a considerable number of mathematical papers which his friend Sir Charles Scarborough had revised under his direction and published at Oxford in 1676 in one volume under the title, *Gulielmi Oughtredi, Etonensis, quondam Collegii Regalis in Cantabrigia Socii, Opuscula Mathematica hactenus inedita.* Its nine tracts are of little interest to a modern reader.

Here we wish to give our reasons for our belief that Oughtred is the author of an anonymous tract on the use of logarithms and on a method of logarithmic interpolation which, as previously noted, appeared as an "Appendix" to Edward Wright's translation into English of John Napier's *Descriptio,* under the title, *A Description of the Admirable Table of Logarithmes,* London, 1618.

The "Appendix" bears the title, "An Appendix to the Logarithmes, showing the practise of the Calculation of Triangles, and also a new and ready way for the exact finding out of such lines and Logarithmes as are not precisely to be found in the Canons." It is an able tract. A natural guess is that the editor of the book, Samuel Wright, a son of Edward Wright, composed this "Appendix." More probable is the conjecture which (Dr. J. W. L. Glaisher informs me) was made by Augustus De Morgan, attributing the authorship to Oughtred. Two reasons in support of this are advanced by Dr. Glaisher, the use of *x* in the "Appendix" as the sign of multiplication (to Oughtred is generally attributed the introduction of the cross × for multiplication in 1631), and the then unusual designation "cathetus" for the vertical leg of a right triangle, a term appearing in Oughtred's books. We are able to advance a third argument, namely the occurrence in the "Appendix" of (S*) as the notation for sine complement (cosine), while Seth Ward, an early pupil of Oughtred, in his *Idea trigonometriae demonstratae,* Oxford, 1654, used a similar notation (S'). It has been stated elsewhere that Oughtred claimed Seth Ward's exposition of trigonometry as virtually his own. Attention should be called also to the fact that, in his *Trigonometria,* page 2, Oughtred uses (') to designate 180°-angle.

COLORADO COLLEGE. FLORIAN CAJORI.

BERGSON'S THEORY OF INTUITION.

Probably the best example of Bergson's application of the intuitive method is to be found in his account of the ideal genesis of the intelligence in the third chapter of *Creative Evolution.* This gives us the gist of his whole philosophy, and serves to illustrate the difficulties of Bergson's view not only of the nature of intellect, but also of intuition itself. What Bergson proposes to do is "to engender intelligence, by setting out from the consciousness which envelopes it"; that is to say, he proposes that we should actually experience in our own selves the process by which duration, which is pure heterogeneity and pure activity, is degraded into the spatializing intellect and spatialized matter. The intellect left to itself, Bergson argues, naturally tends to the homogeneous and the extended and the static. That is to say, the impression we get of the intellect is as of something unmaking itself. "Extension appears only as a tension which is interrupted." But this suggests to us a

reality of which the intellect is merely the degradation and suppression. "The vision we have of the material world is that of a weight which falls; no image drawn from matter, properly so called, will ever give us the idea of the weight rising." But in the case of life we see "an effort to mount the incline that matter descends." Living things "reveal to us the possibility, the necessity even, of a process the inverse of materiality, creative of matter by its interruption alone" (p. 259), a reality which is purely active, a cosmic impulse which makes itself incessantly.

Now if by means of a powerful effort of the mind we succeed in attaining to this reality, if, as Bergson expresses it, "we put back our being into our will, and our will itself into the impulsion it prolongs, we understand, we feel, that reality is a perpetual growth, a creation pursued without end" (p. 252). But if then we relax the tension which this effort demands, we shall ourselves see, or rather *be,* the reverse movement by which the cosmic impetus is degraded, by a kind of process of solidification or chilling or crystalization, into matter and intellect. Reality is pure creative activity, but apparently this creative activity is interrupted or diverted, and in this interruption of the creative current the material world and the spatializing intellect arise. But the creative current is not degraded utterly nor all at once. It still retains even in its degradation some of the force of the main cosmic stream from which it has been diverted. And so the material world and the materialized and materializing intellect, short apparently of pure mathematics and the mathematical intellect, always exhibit two contrary movements. Matter tends naturally toward homogeneous space and necessary determination, just as the intellect left to itself tends toward geometry. But nevertheless this movement is always counteracted by some form of life the function of which is always to convert determination into indetermination and liberty.

By means of this theory, Bergson thinks, it is possible to avoid the difficulty which confronts the Kantian philosophy as to how it has come about that the categories are adapted to work upon the manifold of sensibility at all. Kant had supposed "that there are three alternatives, and three only, among which to choose a theory of knowledge: either the mind is determined by things, or things are determined by the mind, or between mind and things we must suppose a mysterious agreement. But the truth is that there is a fourth alternative which consists first of all in regarding the intellect as a special function of the mind, essentially turned toward

inert matter; then in saying that neither does matter determine the form of the intellect, nor does the intellect impose its form on matter, nor have matter and intellect been regulated in regard to one another by we know not what pre-established harmony, but that intellect and matter have progressively adapted themselves one to the other in order to attain at last a common form. *This adaptation has, moreover, been brought about quite naturally, because it is the same inversion of the same movement which creates at once the intellectuality of mind and the materiality of things*" (p. 217).

This theory seems to raise far more difficulties than it solves. In the first place it is difficult to understand how the cosmic impulse ever can become degraded at all. Is it because the cosmic impulse, which is God, unceasing life, action and freedom, becomes weary? If so, what becomes of the argument that the cosmic impulse is pure creative activity?—an argument which alone, according to Bergson, can save us from the difficulties and deadlocks of the intellect. The metaphor of the stream of life which becomes diverted by matter only to get a better grip on matter does not help in the least, because this theory was put forward as explaining the genesis of matter. Instead of pure duration explaining matter, matter has to be appealed to in order to explain duration.

Moreover there is a further difficulty in this account of the ideal genesis of matter in connection with Bergson's view of the nature and validity of mathematics. Matter is constituted by the reversal of the cosmic impetus, but this movement of matter toward externality and spatiality is never complete. "Matter is extended without being absolutely extended," because in every actual material system there is always a certain amount of interaction between the parts, whereas in a purely extended system every part would be utterly indifferent to every other part. But although the reversal of the cosmic impetus has originated at once "the intellectuality of mind and the materiality of things," yet the intellect outruns the spatiality of things, and so we get pure mathematics. If this is so then it is untrue to say, as Bergson does, that "intellect and matter have progressively adapted themselves one to another to attain at last a common form" (*Creative Evolution*, p. 217), and we have not bridged over the Kantian antithesis of matter and form.

"Our perception," Bergson says, "whose rôle it is to hold up a light to our actions, works a dividing up of matter that is always too sharply defined, always subordinate to practical needs, consequently always requiring revision. Our science, which aspires to

the mathematical form, over-accentuates the spatiality of matter; its formulas are, in general, too precise, and ever need remaking" (p. 218). "Laws mathematical in form can never be applied completely to matter; for that matter would have to be pure space, and to separate itself from duration." "One cannot insist too much on the artificial element in the mathematical form of a physical law, and consequently in our scientific knowledge of things." "Physics comprehends its rôle when it pushes matter in the direction of spatiality." This becomes still more puzzling when we find that although mathematics pursues this process of further falsifying the false product of the intellect, yet at the finish mathematics gives us "a veritable means of contact" with the Absolute.

The whole argument seems to reduce to this: The practical life, which is, so to speak, a smaller stream diverted from the cosmic impulse in which reality consists, is occupied with the penetration and utilization of material things in order to overcome their determination. In order to help the practical life to realize this object the intellect misrepresents the nature of material things. Mathematics goes one better and carries out completely that process of falsifying reality for the sake of which the intellect was created by life, but in so doing mathematics succeeds in attaining to that reality of which not only material things, but even life itself, are mere degradations. And, further, this reality has degraded itself into intellect and matter in order to overcome the resistance of those falsities, with which it is the business of the intellect to provide life, in order to help life to overcome these falsities.

This is an argument which simply makes one giddy. And it is an argument which shows quite conclusively that Bergson can get meaning into his intuition only by appealing to intellect. Instead of intuition explaining intellect it is always intellect which is used to explain intuition. In answer to this it does not help at all to say, as Bergson does, that matter is always "ballasted with geometry." Matter is ballasted with geometry because our intellect tends naturally toward mathematics. But if our intellect tends towards mathematics, so much the worse, on Bergson's doctrine, for our intellect. Bergson has really repeated the performance of Kant. He has offered a justification of mathematics which is really the condemnation of mathematics. If the only reality is intuition then mathematics is false. And if, on the other hand, mathematics is true then the intellect and not intuition gives truth.

To sum up. All of Bergson's arguments for the incompetence

of the intellect break down. They either beg the question at once, in that intellect is defined in such a way as to mean something which cannot give truth, or they involve an appeal to the intellect the incompetence of which they are designed to demonstrate. On the other hand, Bergson's intuition, which is to give relief from the deadlocks which the intellect creates, will do nothing of the sort. Not only does it raise more difficulties than it solves, but it can only be expressed by reference to the intellect and the objects of the intellect.

The reason for this collapse of the Bergsonian philosophy is obvious. Bergson has only repeated the mistakes for which he reproaches Kant. In order to save the freedom of the will, God and immortality from all possible assaults of the intellect, Kant put these realities outside all possible knowledge. In much the same way, in order to have an answer to all possible difficulties which the intellect creates (and because he is apparently ignorant of the intellectual solution of certain classic difficulties), and in order to be able to say that what gives truth is not intellect, Bergson has to make intellect and intuition radically opposed to one another. But having so separated intellect and intuition, Bergson cannot justify either of them. He has not dealt fairly with intellect and has restricted it beforehand to that which is assumed not to be real. At the same time every attempt which Bergson makes to apply his doctrine of intuition, to show why intuition is necessary to supplement intellect, contradicts his own account of the nature of intuition.

E. H. Strange.

University College, Cardiff, Wales.

ANYNESS AND PURE FORM.

On another page in this issue, Prof. E. H. Strange criticizes Bergson's theory of intuition and derives it from Bergson's opposition to Kant's idealism.

In this connection it is appropriate to state that Kant bases his philosophy on the consideration that the highest laws of nature are identical with the mathematical or purely formal theorems. The latter are verified and indeed created by pure reflection, which means they are mind-made; or, as Kant expresses himself, they are products of *a priori* thought, they are transcendental, they serve us as the forms with whose help we reduce sense-impressions to well regulated experiences.

Kant asks in his Prolegomena,[1] "How is the agreement between the highest laws of nature and the theorems of purely formal thought possible?" and he sees only two possibilities. Either, says Kant, we find these laws in nature by experience or the mind makes these laws, and his answer is that the human intellect is so constituted that it can see the world only as its own tools shape it. Kant declares that the world of material things surrounding us can be recognized by the mind only according to the mind's constitution, not as the things are in themselves. The mind imposes its own laws upon the objective world. The opposite view, that the mind has derived its laws from the objective world, is excluded because we know positively that mathematics are mind-made, they are *a priori*. We can construct all mathematics without appealing to any experience of the senses.

Crusius, a German contemporary of Kant, proposed the theory that some world intellect, the creator or God, has established a preconceived harmony between mind and the universe, equipping the mind of man with such a mentality as to enable him to build up the highest (the purely formal) laws of the world constitution out of his own mental resources—a proposition which is quite plausible before a tribunal of theologians, but scarcely acceptable to philosophers.

Now comes Bergson, and having gone through a study of Kant (according to Professor Strange) he finds himself nonplused by Kantian idealism, and he sees another, a fourth, way out of the dilemma. On the basis of a misconceived interpretation of evolution he proposes that "intellect and matter have progressively adapted themselves to one another to attain at last a common form" (*Creative Evolution*, p. 217). This fourth possibility as proposed by Bergson is probably the most unfortunate theory of all, for it presupposes the notion that neither the highest laws of nature nor the truths of mathematical propositions have been stable.

Bergson seems to assume that the highest laws of nature as well as mathematical theorems were loose rules in the beginning and have gradually hardened into definite norms. The intellect and the material world have been in contact and have influenced each other. Our observation of the stars has gradually impressed itself upon their movements so as to assume more and more a definite mathematical form. Finally Kepler succeeded in sum-

[1] This and the following quotations are from memory.

marizing their motions in definite mathematical formulas. Before the mind was in touch with them they may have had other uniformities, or lack of uniformity, of motion. On the other hand the mind was rambling at first and mathematical theorems varied; but gradually they assumed definite form, and now a thinking being can evolve them out of the resources of his own mind by *a priori* argument.

This kind of interpretation of the agreement between mind and nature by a mutual adaptation of the intellect on the one side and the objective world on the other, displays a lack of insight into the very nature of mathematics, and misconceives also the character of natural law.

Take for instance the simple *a priori* statement that $2 \times 2 = 4$. Can there ever have been a time in which this statement was not true? There was a time indeed when the mind could not think in figures at all, when an arithmetical equation or an algebraic formula or a geometrical theorem must have been unmeaning to a sentient being. Indeed formal thoughts are still void of meaning to animals and are above the comprehension of savages; nevertheless their truth is established, and the celestial bodies moved according to the laws of Kepler before mankind originated and mathematical theorems were ever constructed. Kepler discovered his three laws; he did not invent them. To think that the objective truth of the highest laws of nature originated through a process of evolution indicates a misunderstanding not only of the very nature of mathematics, but also of the theory of evolution, and finally also of science itself.

We believe that Kant raised the problem of problems in philosophy, and explained his reasoning in his *Prolegomena,* which therefore, in our opinion, is the most important book that came from his pen. Kant's significance and the prominent place he holds in philosophy are due to the fact that he put his finger on the critical question, though he did not succeed in answering it. He established beyond the shadow of a doubt the apriority of all the formal sciences, but he explained this truth wrongly and has thus given rise to a wrong idealism, deriving therefrom an agnosticism which he formulated in the doctrine that things in themselves are unknowable. His disciples have come to the conclusion that things in themselves do not exist, and we suggest that what he really meant were "forms in themselves" viz., the Platonic ideas or types of things and they are not unknowable.

In consideration of the significance of the Kantian problem, which was suggested to him by Hume's skepticism as to the universal validity of the law of causation, we have published a translation of Kant's *Prolegomena* with our own criticism, substituting for Kant's solution our own which is the basis of the philosophy of science.

We grant that all formal knowledge, including logic and mathematics, is *a priori*, but the conditions for a construction of mathematics after all presuppose experience and the basis of mathematics is the creation of an abstract realm of pure form. For a construction of the purely formal sciences we exclude everything particular and concrete, matter as well as energy, and retain only our own activity with a scope of pure motion which involves the possibility of constructing pure interrelations. In other words, the tools with which we operate are ultimately derived from experience. We retain our ability to operate, our activity, our mode of moving about, but we move in a field void of particularity, a field which therefore can be applied anywhere.

We insist that mathematics and all the other purely formal sciences are not constructed from nothingness; they are ultimately based on experience. But from this experience is excluded everything that pertains to sense-experience, and we produce in this way a domain in which we construct relations that do not contain particulars, but outline conditions which apply anywhere to any place and to any time, and we have called this field of pure thought "anyness." The very term anyness contains an explanation of why these propositions can be applied anywhere, and this application anywhere involves that *a priori* propositions are both (as Kant rightly declares) universal and necessary.

By understanding the full significance of anyness, we understand also that these laws of pure form must apply to any possible world, real or imaginary. Thus we can in pure thought deduce the inevitable results of conditions under any circumstances, and we can understand that if there is a world of concrete materiality, its motions, constructions, formations and results of any kind of actions are—so far as their forms, their relations, are concerned—predetermined by the laws of pure form, viz., by the laws of anyness.

Thus harmony must obtain between the purely formal laws as we have produced them by *a priori* construction and as they appear in the concrete world of reality, because the two are the same. Suppose two mathematicians construct a parabola with the same co-

ordinates, would they not both come to the same conclusions as to the form of the parabola? And suppose that in reality a comet is determined by forces which possess a one-to-one correspondence with these same coordinates, would not the path of the comet possess a one-to-one correspondence to the figures of the parabolas of the two astronomers? The determinedness of all purely formal constructions is truly universal and applies anywhere in the domain of mathematics or pure thought and in any possible real world, also in this our world, i. e., the universe in which we live.

Considering the immanent necessity of the laws of form we can understand that this pre-established harmony has not been made by some supernatural being nor can it have originated gradually by a process of evolution, but it is intrinsically necessary. It is the immanent order which is the condition both of our natural laws and the intelligibility of existence. It is that same intrinsic regularity which can be observed everywhere in nature. This same regularity in the domain of form makes it possible that rational beings originate, that science can be established, that ideals can be proposed and lived up to, that a code of morality and a norm of right conduct can be formulated, and that the universe presents itself as a well-regulated and law-ordained cosmos.

A revision of almost all problems of philosophy from our standpoint will shed new light on their solutions, as will appear when we consider Prof. Hartley B. Alexander's article on "The Definition of Number." When enumerating the different conceptions of the interrelation between logical and mathematical views on the one side and philosophy on the other, he omits to mention the solution offered by the philosophy of form, which alone can be regarded as the philosophy of science.

Mr. Bertrand Russell sees the most essential feature of mathematics in its logical interrelations and goes so far as to claim that mathematics has nothing to do with space. Without objecting to definitions we prefer to regard at least geometry as the purely formal science of extension, which means space, not real space but pure or mathematical space. Mathematics presupposes logic and contains one additional element which is commonly called space, but like all purely formal sciences mathematics produces its objects of investigation by *a priori* construction. The elements with which we start are products of abstract thought in the realm of pure form. created by thinking away everything that is particular, viz., all concrete objects that consist of matter and energy. Thus we retain

the idea of pure motion and a possibility of establishing pure interrelations.

Pure motion means a change of place without implying energy, and a possibility of pure interrelations is a field of pure motion. We start with these two abstract notions, on the part of the subject an ability to move about, on the part of the object, (i. e., the surrounding world), emptiness; and this emptiness offers a field of possible motion. With these conditions we construct whatever we may be pleased to build up, and observe the result.

In geometry we do something and note what will come of it. For instance, we move and note the trace of our motion. We call it a line. We move again and again, and let the traces of other lines enclose a space; we call the result a figure. Where two lines cross we have a point.

The system under construction may be Euclidean or non-Euclidean according to our start, whether or not we assume we are able to draw straight lines in the Euclidean space.[3] If in our plan of construction we exclude the straight line, we will have to move according to a definite principle in curves of a predetermined constant deviation, in which case our system will be different from the system of Euclid.

If two straight lines cross, the product of our construction is an angle, or rather four angles. The peculiarity of mathematics is to watch and observe the inevitable results of our own constructions, but the main characteristic of our constructions is this, that they are made in a field of anyness, i. e., they apply to any kind of construction made in the same way, not only in emptiness, but in any kind of a world filled with any kind of matter or any kind of energy.

The nature of matter and energy can only be discovered by experience through the senses, but the nature of pure interrelations can be determined by building up constructions in a field of anyness, as they must be under any conditions, which means under all conditions. Therefore the laws of pure form (in other words, the laws of anyness) will be valid for any kind of a world.

Thus we have an explanation why the theorems of pure mathematics are hyperphysical truths, and here we have a specimen of the nature of what theology has called the supernatural. There is only this difference between the old conception of the supernatural and this new conception of it which for the sake of distinction

[3] For an *a priori* construction of the plane, the straight line and the right angle see the author's *Foundations of Mathematics.*

we call the "hyperphysical," that the latter is as clear and self-evident as the former is mysterious, hazy, bewildering and mystifying.

The consequence of this conception of mathematics need not be traced here in all details, but we feel assured that in the long run it will solve all the modern problems of philosophy and dispose of the troubles which have been caused by pragmatism, Bergsonianism, by the advocates of the principle of relativity, and also by the logisticians.

EDITOR.

LOUIS COUTURAT (1868-1914).

Besides the carnage in battleships and trenches, the great European war carries with it many accidental by-products of disaster not to be overlooked when casting up the grand total of losses the world is suffering. In the early days of last August when the first commotion in the commercial arteries to and from Paris was at its height, a heavy automobile at full speed chanced to run down the carriage in which Louis Couturat was traveling, and his immediate death was the result. Though only forty-six years old he held first rank in France among scientific workers in the philosophy of language, the philosophy of mathematics, and especially in the more modern aspect of logic—for which he agrees with English logicians in preferring the term "logistic," now that this word is but little known in its earlier significations listed in the dictionaries.

M. Couturat was singularly well informed on many questions, but the particular power and quality of his mind lay in a gift for deductive reasoning combined with the most punctilious intellectual honesty that would never countenance a compromise with the truths of reason. All his work is especially remarkable for the clearness of its representation. His style is never sullied by glittering and bizarre phrases intended to attract attention and admiration, but which often seem to cover a multitude of sins in the way of vague ideas and loose reasoning.

Couturat was first known by his painstaking and illuminating exposition of the mathematical infinite (*L'infini mathématique*, 1896) in which he discusses the idea of number and analyzes the concepts of continuity and the infinite, refuting practically all of Renouvier's arguments against the latter. His research in this line familiarized him with all the writings of Leibniz, and his next published work was an edition of more than two hundred fragments

from Leibniz's unpublished manuscripts, some of which proved to be of the greatest philosophical interest. This was followed by a scholarly work on Leibnizian logic (*La logique de Leibniz,* 1901).

It was through their common interest in Leibniz that Couturat became acquainted with the Hon. Bertrand Russell in England, whose *Philosophy of Leibniz* appeared at this time, and their relation continued to be of the friendliest. Couturat added some notes to Cadenat's French translation of Russell's *Principles of Geometry* and introduced his *Principles of Mathematics* to the French public through a series of articles later collected into a book. Readers of *The Monist* will remember his answer to Poincaré's witty sallies against logistics in the issue of October, 1912. In an introduction to this article, M. Couturat's translator, Mr. Philip E. B. Jourdain, summed up the controversy between these two brilliant Frenchmen.

In the meantime, Couturat had published his *Algèbre de la logique.* In a small monograph of less than one hundred pages he presents a concise outline of the material contained in the first two volumes of Schröder's prolix three-volumed treatise. He follows Schröder in making the notion of inclusion the fundamental notion in his calculus in preference to the idea of equality, as the English logicians had done and as Schröder also had done in the beginning, though he made the change later under the influence of C. S. Peirce. Besides brevity Couturat's little work possesses the further advantage of clear-cut precision of argument which makes it practically the most easily intelligible presentation of the subject in any language. It is for this reason that the Open Court Publishing Company only last year issued an English edition of it.

Couturat believed thoroughly in the possibilities and desirability of an international artificial language, and he and Professor Ostwald are the two leading scientific men of whom the Esperanto and Ido movements can boast. In the light of M. Couturat's high character, talents and attainments it can only seem trite and trivial to say that the world has suffered an irreparable loss in his death. L. G. R.

CURRENT PERIODICALS.

The best produced scientific magazine in Great Britain is *Science Progress in the Twentieth Century: A Quarterly Journal of Scientific Work and Thought,* which is edited by the eminent pathologist Sir Ronald Ross. The first article in the number for April 1915 is "Some Aspects of the Atomic Theory" by Frederick

Soddy. "Either matter must occupy space continuously or it must exist in the form of discrete particles. The historical origin of the atomic theory of matter is to be found in the choice between the two possible answers to these mutually exclusive alternatives......" However, "the true origin of the atomic theory is recognized universally to have been during the first decade of the last century in Dalton's discovery of the simple laws of chemical combination, though, even to the discoverer himself, the laws of gaseous behavior, upon which later the totally distinct but inextricably interwoven molecular theory was to be based, undoubtedly played a part in directing the interpretation he put upon these laws. Henceforth science was to deal no longer with atoms as the end results of a purely mental process of the subdivision of matter, a process which must of necessity have an end if matter does not occupy space continuously, but with atoms of definite mass determinable simply and exactly relatively, that is, the mass of any one kind of atom in terms of that of any other." The article, as we should expect, deals with the modern aspects. Francis Hyndman writes on "The Electrical Properties of Conductors at Very Low Temperatures," these properties indicating relations between widely different properties of matter. Arthur E. Everest writes on "The Anthocyan Pigments." The term "anthocyan" now denotes a large class of naturally occurring plant pigments, and the present article contains a very valuable account of the advances in this field of research from 1836 up to the present time. Richard Lydekker contributes a summary of "Vertebrate Palæontology in 1914." "The most important part of the year's work is undoubtedly that on the mammal-like reptiles and their structural resemblances and relationships." Charles Davison deals with "The Prevision of Earthquakes." "Between foreseeing and foretelling an unexpected event, there would seem to be little if any difference, beyond the fact that the one may be conducted in private while the other implies publication of some kind. But, to the corresponding words 'prevision' and 'prediction,' somewhat different meanings seem to be attributed, prevision being apparently considered as an approximate, and prediction as an accurate, form of forecast." This distinction is assumed in the present paper which contains a very good review of our knowledge on the subject. James Johnstone has an interesting discussion on "Is the Organism a Mechanism?" The concluding sentence of the article must be quoted here: "It may be, of course, that the activities of the organism are capable of reduction to chemical and

physical processes, all of which are to be regarded as special cases of the second law—in that event biology is only a department of physical chemistry, and our conception of life must be a mechanistic one. But so long as physiology fails to provide physico-chemical explanations of vital processes, and so long as another physics and chemistry than that of the second law [of thermodynamics] is conceivable, then a real science of biology may be possible; and to insist on a mechanistic conception of the organism is only to dogmatize." Besides these articles, the number contains very interesting and long "Essay-Reviews" as well as shorter reviews of scientific books, and also correspondence.

* * *

In the number of "*Scientia*" (*Rivista di Scienza*) for March, 1915, the first article is by Fritz Frech on the saline seas of Anatolia and their importance for the problem of the origin of blocks of salt in the outer surface of the earth. Eugenio Rignano brings to a conclusion his series of articles on the higher forms of reasoning. In this third part, after a summary of his former two parts on the symbolism of mathematics, he compares mathematics with the new mathematical logic, and arrives at the conclusion that, from a psychological point of view, it would be quite a mistake to hope from the symbolism of mathematical logic the immense advantages that the introduction of symbolism has had in mathematics properly so-called. The inquiry upon the war still continues: this number contains an article written in Italian by Vilfredo Pareto of Lausanne, in which an attempt is made to treat the causes of the war from an entirely objective point of view; William J. Collins has an article in English discussing the deeper origins of the war; and Eduard Meyer writes in Garman on "England's War Against Germany and the Problems of the Future." These articles, except perhaps the last, have a refreshingly scientific air about them. Georges Chatterton-Hill contributes a critical note on Treitschke's *Ausgewählte Schriften*. There are reviews of books and periodicals, and a supplement containing French translations of the English, German and Italian articles.

In "*Scientia*" for April, 1915, the first article is by Aldo Mieli on the position of Lavoisier in the history of chemistry, in which it is brought out that Lavoisier ended a period instead of beginning one. J. W. Gregory writes on "The Reported Progressive Desiccation of the Earth," and finds no reason for believing that the earth

is approaching a world-wide drought. The splendid scheme of "The Inquiry upon the War" continues: in this number we have articles by N. Kostyleff of Petrograd on the psychological factors of the war, by L. M. Hartmann of Vienna University on the causes of the war and by Lujo Brentano of Munch University on the deepest causes of the war. Of these three articles, the most interesting is undoubtedly the first very broad-minded study, writen from the laboratory of pathological psychology of the "Ecole pratique des Hautes Etudes" of Paris. There are the usual reviews of books and periodicals, a chronicle, and French translations of the Italian, English, and German articles. Φ

EDITORIAL COMMENT.

The opinion here expressed of the value of the *Scientia* contributions on the war represents the judgment of the English reviewer. We will only add that Eduard Meyer is a prominent historian who received the degree of doctor *honoris causa* from the University of Chicago. Writing from a German standpoint he naturally holds the English government responsible for the cause of the war. He enumerates his reasons in clear and terse language. In conclusion he predicts that unless the sea shall become equally free to all nations this war will be the beginning of further wars, and that an incidental but important result will be the unexpected growth of Japanese power and a gigantic struggle for supremacy in the Pacific and Indian oceans. Professor William J. Collins contrasts two world-conceptions, one is "science-ridden," "materialistic," in which "the state displaced the church," "matter and force are the masters," "disinterested virtue and sympathetic compassion are sacrificed to the will to power," "the will....heart, conscience, soul....[are] dismissed as so much metaphysical moonshine," "a brand new religion for Supermen." The nation that fits this description is not named but may easily be guessed if we bear in mind that the author is an Englishman who calls these curious comments "The Aetiology of the European Conflagration." This discussion of "the deeper origin of the war" has indeed "a refreshingly scientific air" about it. At any rate Professor Eduard Meyer will find this scientific conception of history refreshing. P. C.